Annual Report to the Air Force Office
of Scientific Research

Nov. 1, 1988 - Oct. 31, 1989

DYNAMICS AND CONTROL OF TETHERED ANTENNAS/ REFLECTORS IN ORBIT

Contract No. F49620-89C-0002

Principal Investigator: Peter M. Bainum
Professor of Aerospace Engrg.
Dept. of Mechanical Engrg.

Visiting Scholar: Liu Liangdong
Dept. of Mechanical Engrg.

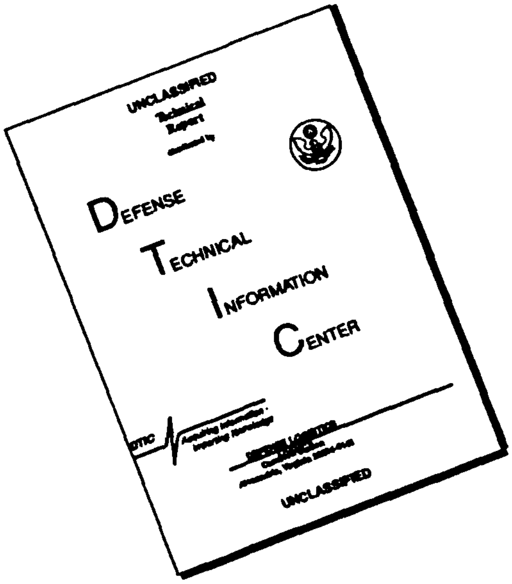
Visiting Scholar: Bai Jingwu
Dept. of Mechanical Engrg.

Graduate Research Assistant: Li Zhong
Ph.D. Student
Dept. of Mechanical Engrg.

Howard University
School of Engineering
Washington, D.C. 20059

December 1989

DISCLAIMER NOTICE



THIS DOCUMENT IS BEST QUALITY AVAILABLE. THE COPY FURNISHED TO DTIC CONTAINED A SIGNIFICANT NUMBER OF PAGES WHICH DO NOT REPRODUCE LEGIBLY.

Annual Report to the Air Force Office
of Scientific Research

Nov. 1, 1988 - Oct. 31, 1989

DYNAMICS AND CONTROL OF TETHERED ANTENNAS/ REFLECTORS IN ORBIT

Contract No. F49620-89C-0002

Principal Investigator:	Peter M. Bainum Professor of Aerospace Engrg. Dept. of Mechanical Engrg.
Visiting Scholar:	Liu Liangdong Dept. of Mechanical Engrg.
Visiting Scholar:	Bai Jingwu Dept. of Mechanical Engrg.
Graduate Research Assistant:	Li Zhong Ph.D. Student Dept. of Mechanical Engrg.

Howard University
School of Engineering
Washington, D.C. 20059

December 1989

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE

REPORT DOCUMENTATION PAGE

1a. REPORT SECURITY CLASSIFICATION
Unclassifed

1b. RESTRICTIVE MARKINGS

2a. SECURITY CLASSIFICATION AUTHORITY
-

2b. DECLASSIFICATION/DOWNGRADING SCHEDULE
-

3. DISTRIBUTION/AVAILABILITY OF REPORT
Approved for public release; distribution is unlimited

4. PERFORMING ORGANIZATION REPORT NUMBER(S)
-

5. MONITORING ORGANIZATION REPORT NUMBER(S)
-

6a. NAME OF PERFORMING ORGANIZATION
Howard University

6b. OFFICE SYMBOL (If applicable)
-

7a. NAME OF MONITORING ORGANIZATION
Air Force Office of Scientific Research
[redacted]

6c. ADDRESS (City, State and ZIP Code)
Dept. of Mechanical Engineering
Washington, D.C. 20059

7b. ADDRESS (City, State and ZIP Code)
AFOSR/NA
[redacted] Bolling AFB DC 20332-6448

8a. NAME OF FUNDING/SPONSORING ORGANIZATION
[redacted] AFOSR

8b. OFFICE SYMBOL (If applicable)
- NA

9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER
F49620-89C-0002

8c. ADDRESS (City, State and ZIP Code)
AFOSR/NA
[redacted] Bolling AFB DC 20332-6448

10. SOURCE OF FUNDING NOS.

PROGRAM ELEMENT NO.	PROJECT NO.	TASK NO.	WORK UNIT NO.
61102F	2302	B1	--

11. TITLE (Include Security Classification) Dynamics and Control of Tethered Antennas/Reflectors in Orbit (Unclass.)

12. PERSONAL AUTHOR(S)
Dr. Peter M. Bainum, Liu Liangdong, Bai Jingwu, and Li Zhong

13a. TYPE OF REPORT	13b. TIME COVERED	14. DATE OF REPORT (Yr., Mo., Day)	15. PAGE COUNT
Final	FROM 11-1-88 TO 10-31-89	December 1989	85

16. SUPPLEMENTARY NOTATION

17. COSATI CODES

FIELD	GROUP	SUB. GR.

18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number)
Dynamics and control; orbiting reflector/antenna systems; vibration suppression. (EG)

19. ABSTRACT (Continue on reverse if necessary and identify by block number)

A study is conducted of the dynamics and control of a class of large antenna/reflector systems in orbit which are also partially stabilized using a tether-connected subsatellite. The system equations of motion are developed and linearized about the equilibrium position where the reflector's (shell's) symmetry axis nominally follows the local vertical. The shell roll, yaw, tether out-of-plane swing motion and out-of-plane elastic vibrations are decoupled from the shell and tether in-plane pitch motions and in-plane elastic vibrations. The in-plane motion of the system could be asymptotically stable based on Rupp's tether tension control law using only length and length rate information. However, the transient responses can be improved significantly by using an optimal tension feedback control law. When tether flexibility is included, the system dynamics could be further improved by including the state feedback of the tether vibrational modes into the tension control law. A literature survey including a brief comparison of different proposed control laws is presented. Finally, a preliminary model of the nonlinear dynamics has been obtained based on Lagrangian techniques.

20. DISTRIBUTION/AVAILABILITY OF ABSTRACT
UNCLASSIFIED/UNLIMITED ☒ SAME AS RPT. ☐ DTIC USERS ☐

21. ABSTRACT SECURITY CLASSIFICATION
Unclasified

22a. NAME OF RESPONSIBLE INDIVIDUAL
Col. George K. Haritos

22b. TELEPHONE NUMBER (Include Area Code)
(202)767-0463

22c. OFFICE SYMBOL
ATCH 1

DD FORM 1473, 83 APR

EDITION OF 1 JAN 73 IS OBSOLETE.

SECURITY CLASSIFICATION OF THIS PAGE
UNCLASSIFIED

SUMMARY

The purpose of the reported research is to study the dynamics and control of a class of large antenna/reflector systems in orbit which are also partially stabilized using a tether-connected subsatellite. The initial focus has been in the development of the system's equations of motion linearized about the equilibrium position where the reflector's (shell's) symmetry axis nominally follows the local vertical. The shell roll, yaw, tether out-of-plane swing motion and out-of-plane elastic vibrations are decoupled from the shell and tether in-plane pitch motions and in-plane elastic vibrations. It is proved that the in-plane motion of the system could be asymptotically stable based on Rupp's tether tension control law based only on length and length rate information. However, the transient responses can be improved significantly (especially for damping of the tether and shell pitch motion) by using an optimal tension feedback control law. When tether flexibility is included tension control law gains must be carefully selected in order to preserve stability. System transient responses could be further improved by including the state feedback of the tether vibrational modes into the optimal tension control law.

In order to prepare for an extension of this study to simulate the deployment or retrieval dynamics, a literature survey including a brief comparison of control laws proposed

by different investigators has been completed. Recommendations are made concerning the suitability of the various control laws for use with the orbiting tethered reflector system.

Finally a preliminary model of the nonlinear dynamics of the tethered antenna/reflector system in orbit has been obtained based on Lagrangian techniques. It is seen that, unlike the situation for the system linearized about the nominal stationkeeping motion, the in-plane and out-of-plane motions are coupled through second order, and nonlinear coupling terms also depend on tether line swing motions and tether vibrations. For this preliminary model the shell is considered to be a rigid structure.

ACKNOWLEDGEMENT

This research has been conducted under the direction of Dr. Anthony K. Amos, Program Manager, Aerospace Sciences, Air Force Office of Scientific Research, Bolling AFB, Washington, D.C. Appreciation is expressed for the strong encouragement and useful comments provided by Dr. Amos during the course of this study. During the final phase of this work, Dr. Amos was replaced by Lt. Col. Dr. George Haritos. Thanks are also extended to Dr. Haritos for his support and for making time available for a brief oral progress report. Finally, the interest of Dr. Michael J.Salkind, Director of Aerospace Sciences of the Air Force Office of Scientific Research is also acknowledged.

TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

LIST OF TABLES

1. INTRODUCTION

1.1 Feasibility of Concept based on Existing Work

Since the early 1970's a number of very large space antennas have been proposed for power transmission, astronomical research and communications. The gravity stabilized configuration is particularly suited for a very large flexible structure to alleviate the problems associated with the active attitude control of very large structures. The structural feasibility of a very large Earth oriented antenna, where the flexible reflector contour is maintained by adjusting the length of connecting tethers between the reflector and feed panels, has been discussed.[1] In this paper the stress analysis of the tethered antenna was given. The analysis of the dynamics and control of the orbiting flexible shallow spherical shell and various tether connected systems in space have been performed. Bainum and Kumar[2] have investigated the dynamics of an orbiting flexible shallow spherical shell with a dumbell connected to the shell at its apex by a spring-loaded double-gimball joint to provide the favorable composite moment of inertia distribution. Also, Bainum and Reddy[3] have investigated the shape and orientation control of this shell antenna by including some additional active control elements. Numerical results verify that a significant savings in fuel consumption can be realized by using the hybrid shell-dumbell system together with the (active) point actuators.

The purpose of the proposed research is to study the dynamics and control of a class of large antenna/reflector orbiting structures which include an articulated tether connected supporting structure to provide the favorable moment of inertia distribution for over-all gravitational stabilization together with some active actuators. There are two possible proposed subsystems which could provide the connection between the tether and the shell reflector; one involves a spring-loaded doubled-gimballed joint connected to the shell's apex and through which the tether is deployed/retrieved (Fig. 1); the second contains a joint at the end of a rigid boom which is attached to the shell's apex (Fig. 2). Through the end joint the tether would be deployed or retrieved. The tether tension could be used for producing restoring torques on the shell, with natural damping provided in the joint assembly. For the first phase of the study reported here the second subsystem has been taken as the basis for the system model due to the relatively simpler implementation as compared with the double-gimballed joint in the first subsystem.

1.2 Relevance to SDI

Associated with the capability to orient a large flexible antenna/reflector type of device accurately while at the same time maintain the surface shape to within centimeters or even millimeters are many applications in both the military and civilian fields. For example, high energy

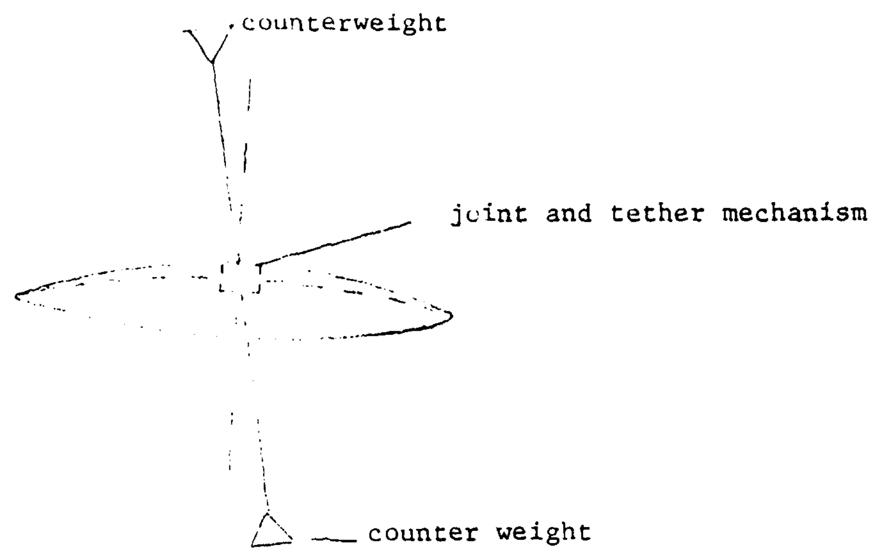


Fig. 1. Subsystem A - Tether Deployed from Apex of Reflector

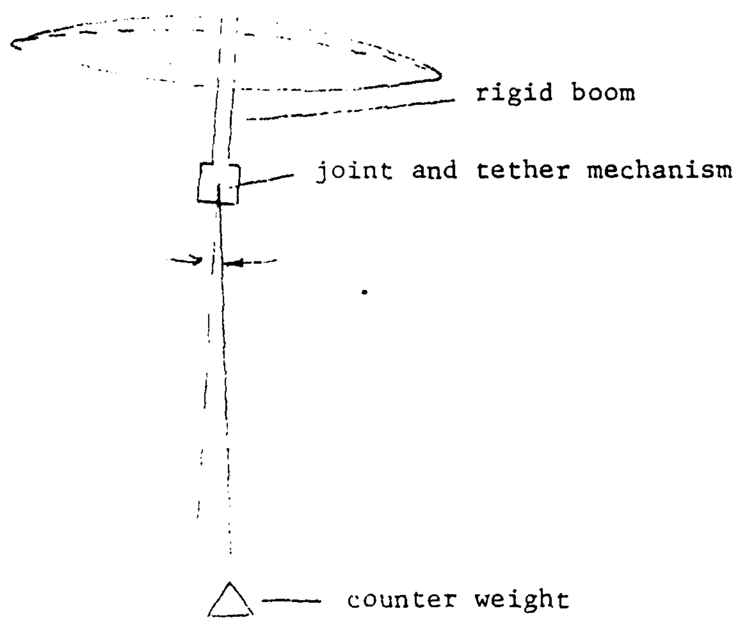


Fig. 2. Subsystem B - Tether Deployed from the End of a Rigid Boom Connected to the Reflector

beams can be generated by a power source and reflected from specific known points on the reflector surface to preselected targets. In the very important communications field, such an antenna surface can receive multibeam communication waves from electronic feed devices and transmit these to a variety of small mobile receivers to comprise strategic communication links during early, critical phases of an attack when larger, fixed land-based antennas would be far more vulnerable to observation/damage. Such devices could also be employed to transmit coded electronic mail rapidly over different communication channels.

1.3 Outline of the Research Reported

The second chapter focuses on the development of the linear system equations of motion for an orbiting tethered shallow spherical shell system where the shell's axis of symmetry nominally follows the local vertical. The Newton-Euler method for a continuous system is adopted here. The second objective is to develop the in-plane and out-of-plane stability conditions and introduce some tension control laws for in-plane motion control. The transient responses will be compared for three different tension control laws during typical station keeping operations. A paper based on these tasks was presented at the Third International Conference on Tethers in Space, San Francisco, Masy 17-19, 1989 and has been accepted in slightly revised format for publication in The Journal of the Astronautical Sciences.

The following chapter describes a comprehensive review of the steps in the development of control laws for Shuttle or platform connected tethered subsatellite systems. Deployment, stationkeeping, and retrieval control strategies are reviewed and compared. Finally, recommendations are made suggesting the relative suitability of the different control laws for adaptation with the proposed orbiting tethered reflector systems.

Chapter Four concentrates on the development of the nonlinear equations of motion for the tethered reflector system in orbit in a form suitable for simulasting deployment and/or retrieval maneuvers based on some of the control laws described in Chapter Three.

Finally, Chapter Five summarizes some concluding statements and follow-on plans for the continuation of this general area of research.

2 DYNAMICS AND CONTROL OF A TETHERED ANTENNA/REFLECTOR IN ORBIT

2.1 Introduction

Since the early 1970's a number of very large space antennas have been proposed for power transmission, astronomical research and communications.[1] The gravity stabilized configuration is particularly suited for very large flexible systems to alleviate the problems associated with the active attitude control of very large structures. Bainum and Kumar[2] have investigated the dynamics of an orbiting flexible shallow spherical shell with a dumbbell connected to the shell at its apex to provide the favorable composite moment of inertia distribution. Also, Bainum and Reddy[3] have investigated the shape and orientation control of this shell antenna by including some additional active control elements. Meanwhile, scores of applications of tethers in space have been proposed and analyzed including some space platform-based applications of the tether subsatellite system.[4]-[5]

The objective of the present paper is, first, to develop a system mathematical model of a class of large antenna/reflector orbiting structures which include an articulated tether-connected supporting structure to provide the favorable moment of inertia distribution for over-all gravitational stabilization, together with some active actuators. The tether would be connected at the end of a rigid boom which is attached to the shell's apex and through the end of the boom the tether could be deployed or retrieved (Fig. 3). The tether tension could be used for producing restoring torques on the shell. The second objective is to develop the in-plane and out-of-plane stability conditions and introduce some tension control laws for in-plane motion control. The transient responses for the three different tension control laws will be compared during typical station keeping operations.

2.2 Equations of Motion

For system modelling the following assumptions were made:

1) The thickness of the shell is small as compared to the height of the shell, and the ratio of the height to the base radius is much less than unity (condition for shallowness).

2) The elastic deformations perpendicular to the symmetry axis(i.e.,x axis) of the shell are negligible compared with the deformations parallel to the symmetry axis, i.e., only transverse vibrations are considered.

3) The symmetry axis of the shell is nominally along the local vertical.

4) The center of mass of the system is moving in a circular orbit.

5) The flexibility of the boom is neglected.

6) The subsatellite is to be considered as a point mass.

The shift of the center of mass of the system will be considered. In order to develop a general model for the tethered shell system it is assumed that the massive, flexible tether is deploying or retrieving a subsatellite at a distance, , from a point on the shell which is offset by distance h_x, h_y, h_z, along the yaw, pitch, and roll axes, respectively, from the center of mass of the shell, O_s.

Santini[6], Bainum and Kumar[7] have developed a mathematical formulation for a general orbiting flexible body based on the Newton-Euler method and continuum approach. In the present paper this method will be extended to the system composed of two flexible structures (the shell and the tethered subsatellite).

The coordinate systems used in the development of the system equations of motion are shown in Fig. 4. $O_sX_oY_oZ_o$ is an orbit-fixed reference frame centered at the center of the mass of the shell, O_s, with O_sX_o along the local vertical and O_sY_o along the orbit normal opposite to the angular velocity vector. $O_sX_sY_sZ_s$ is an undeformed shell reference frame, R_s, where O_sX_s, O_sY_s, O_sZ_s are the principal axes of the shell. OXYZ is the subsatellite-undeformed tether reference frame, R_t, with OX along the undeformed tether line, where O is the point from which the tether is deploying or retrieving. The coordinates of O in the shell frame, R_s, are h_x, h_y, h_z.

The angles ψ, θ, ϕ are the yaw, pitch and roll angles of the shell, respectively. An Euler angle rotation sequence of: (1) ψ, (2) θ, and (3) ϕ is assumed from the $O_sX_oY_oZ_o$ system to the $O_sX_sY_sZ_s$ system.

The transformations from $O_sX_oY_oZ_o$ to $O_sX_sY_sZ_s$ and from $O_sX_sY_sZ_s$ to OXYZ are assumed to be given by

$$\begin{bmatrix} X_s \\ Y_s \\ Z_s \end{bmatrix} = \begin{bmatrix} c\phi c\theta & s\phi c\psi + c\phi s\theta s\psi & s\phi s\psi - c\phi s\theta c\psi \\ -s\phi c\theta & c\phi c\psi - s\phi s\theta s\psi & c\phi s\psi + s\phi s\theta c\psi \\ s\theta & -c\theta s\psi & c\theta c\psi \end{bmatrix} \begin{bmatrix} X_o \\ Y_o \\ Z_o \end{bmatrix} \quad (1)$$

$$[\ X\ \ Y\ \ Z\ ]^T = T(\alpha\ \gamma)\ [\ X_o\ \ Y_o\ \ Z_o\ ]^T$$

where

$$T(\alpha,\gamma) = \begin{bmatrix} -c\gamma c\alpha & s\gamma & s\alpha c\gamma \\ s\gamma c\alpha & c\gamma & -s\alpha s\gamma \\ -s\alpha & 0 & -c\alpha \end{bmatrix} \tag{2}$$

where c → cosine ( ), s → sine ( )

Consider an elemental mass, dm, whose instancous position vector from the center of the shell, O_s, is $\bar{r}$ (Fig.4). The equation of motion for dm can be written as[7],[8]

$$\bar{a}\ dm = L(\bar{q}) + \bar{f}\ dm + \bar{e}\ dm \tag{3}$$

where $\bar{a}$ = inertial acceleration of dm

$\bar{q}$ = elastic displacement vector of dm

$L(\bar{q})$ = elastic forces acting on dm

$\bar{f}$ = gravitational force per unit mass

$\bar{e}$ = external forces acting per unit mass

The gravity force in the shell frame, R_s, is given by[5],[7]

$$\bar{f} = \bar{f}_o + M\ \bar{r} \tag{4}$$

where $\bar{f}_o$ is the gravity force at O_o expressed in the frame, R_o, and

$$M = \omega_c^2 \begin{bmatrix} 3c^2\phi c^2\theta - 1 & -3s\phi c\phi c^2\theta & 3c\phi c\theta s\theta \\ -3s\phi c\phi c^2\theta & 3s^2\phi c^2\theta - 1 & -3s\phi c\theta s\theta \\ 3c\phi s\theta c\theta & -3s\phi c\theta s\theta & 3s^2\theta - 1 \end{bmatrix} \tag{5}$$

where ω_c is the orbital angular velocity.

The vector equation, (3), can be written in the frame, R_s, as

$$\left[\bar{a}_0-\bar{f}_0+\ddot{\bar{r}}+2\bar{\omega}\times\dot{\bar{r}}+\bar{\omega}\times(\bar{\omega}\times\bar{r})+\dot{\bar{\omega}}\times\bar{r}-M\bar{r}\right]dm-L(\bar{q})-\bar{e}dm = 0 \tag{6}$$

where $\dot{\bar{r}}$, $\ddot{\bar{r}}$ are the velocity and acceleration of dm, respectively, as seen from the frame, R_s, and $\bar{\omega}$ is the angular velocity of the frame R_s.

$$\bar{\omega} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} \dot{\theta}s\phi+\dot{\psi}c\phi c\theta-\omega_c(s\phi c\psi+c\phi s\theta s\psi) \\ \dot{\theta}c\theta-\dot{\psi}s\phi c\theta-\omega_c(c\phi c\psi-s\phi s\theta s\psi) \\ \dot{\phi}+\dot{\psi}s\theta+\omega_c c\phi s\psi \end{bmatrix} \tag{7}$$

It is well known that for some applications, for example, for the tethered Shuttle subsatellite system, the mass of the Shuttle is much greater than that of the tethered subsatellite, so the center of mass of the Shuttle can be considered to be the mass center of the whole system and the shift of the center of mass of the system can be neglected i.e. $\bar{a}_0-\bar{f}_0=0$ in equation (6). However, in our system the shift of the center of mass of the system will be considered and, in general, $\bar{a}_0-\bar{f}_0\neq 0$; it will be calculated in the development.

After projecting equation (6) on the tether frame, R_t, the following is obtained

$$(\bar{a}_0-\bar{f}_0)|_t+T\left[\ddot{\bar{r}}+2\bar{\omega}\times\dot{\bar{r}}+\dot{\bar{\omega}}\times\bar{r}+\bar{\omega}\times(\bar{\omega}\times\bar{r})-M\bar{r}\right]dm-L(\bar{q})|_t-\bar{e}|_t dm=0 \tag{8}$$

where $|_t$ indicates the projection onto the frame, R_t.

The expression of the r of the tether system is different from that of the shell due to relative motion of the tether, so we consider the tethered subsatellite system and the shell, separately.

Tethered subsatellite system

$$\bar{r} = T^{-1}\,\bar{r}_t + \bar{h} + \bar{q}_0 \tag{9}$$

where $\overline{r}_t = (x+u, v, w)$ (10)

is the position vector of dm from O projected onto the frame, R_t, u represents the tether's longitudinal elastic displacement. v, w represent the displacements in the orthogonal directions transverse to the OX axis.

$\overline{h} = (h_x, h_y, h_z)$ is the position vector of point O from the shell's center of mass, O_s.

$\overline{q}_0 = (u_{s}, 0, 0)$ is the shell's elastic displacement vector of the apex of the shell (according to the assumptions there is only elastic displacement along the X_s axis).

Hence,
$$\dot{\overline{r}} = (T^{-1}) \dot{\overline{r}}_t + \dot{\overline{q}}_0 + (\dot{T}^{-1}) \overline{r}_t \tag{11}$$
$$\ddot{\overline{r}} = (T^{-1}) \ddot{\overline{r}}_t + 2(\dot{T}^{-1}) \dot{\overline{r}}_t + (\ddot{T}^{-1}) \overline{r}_t + \ddot{\overline{q}}_0 \tag{12}$$

Let
$$[\omega] = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} \tag{13}$$

According to vector algebra
$$\dot{\overline{\omega}} x \overline{r} + \overline{\omega} x (\overline{\omega} x \overline{r}) - M\overline{r} = [Q] \overline{r}$$

where
$$[Q] = [\dot{\omega}] + [\omega][\omega] - [M] \tag{14}$$

After substitution of equations (11)-(14) into equation (8) there results
$$[(\overline{a}_2 - \overline{f}_2) \cdot + \overline{a}_s] dm - L(\overline{q}) \cdot - \overline{c} \, l_t \, dm = 0 \tag{15}$$

where $\overline{a}_t = \ddot{\overline{r}}_t + T\ddot{\overline{q}}_0 + 2[T(\dot{T}^{-1}) + T[\omega]T^{-1}] \dot{\overline{r}}_t - 2T[\omega]\dot{\overline{q}}_0 +$
$$[T(\ddot{T}^{-1}) + 2T[\omega](\dot{T}^{-1}) + T[Q]T^{-1}] \overline{r}_t + T[Q](\overline{h} - \overline{q}_0) \tag{16}$$

In order to form the system linear equations of motion equation (16) is linearized as follows:

$$\bar{a}_t = (a_{t_x}, a_{t_y}, a_{t_z})^T \tag{17}$$

$$a_{t_x}/\omega_c^2 = \ell'' + u'' - u_{p_0}'' - 3u + 3u_{p_0} - 2w' + (2\alpha' + 2\theta' - 3)x$$

$$+ (3 - 2\theta')h_x + (\phi'' + 2\psi' - 4\phi + \gamma)h_y + (3\theta - \theta'')h_z$$

$$a_{t_y}/\omega_c^2 = v'' + v + (\gamma'' + 4\gamma - \phi'' - 4\phi)x + (\phi'' + 4\phi - 3\gamma)h_x + h_y - (\psi'' + \psi)h_z$$

$$a_{t_z}/\omega_c^2 = w'' + 2(\ell' + u') - 2u_{p_0}' - (\alpha'' + \theta'' + 3\theta + 3\alpha)x$$

$$+ (\theta'' + 3\alpha + 3\theta)h_x + (2\phi' - \psi'' + \psi)h_y - 2\theta' h_z \tag{18}$$

where $(\ )' = d(\ )/d\tau$ and $\tau = \omega_c t$, ℓ is the length of the tether

Shell system

Now consider equation (8) for the shell system

$$\bar{r} = \bar{r}_p + \bar{q}_p = (x_p + u_p, y_p, z_p) \tag{19}$$

$$\dot{\bar{r}} = (\dot{u}_p, 0, 0)\ ;\ \ddot{\bar{r}} = (\ddot{u}_p, 0, 0) \tag{20}$$

where x_p, y_p, z_p are the coordinates of dm of the shell in the frame, R_p.

After substitution of equation (20) into equation (8) there results

$$[(\bar{a}_o - \bar{f}_o)|_t + \bar{a}_p]\, dm - T\, L(\bar{q}_p) - \bar{e}|_t\, dm = 0 \tag{21}$$

where $$\bar{a}_p = [3u_p - \ddot{u}_p, 0, -2\dot{u}_p]^T + T\,[Q][x_p, y_p, z_p]^T \tag{22}$$

Equation (15) is integrated over the tethered subsatellite and equation (21) is integrated over the shell. The two results are added together and it is obtained that

$$(\bar{a}_o - \bar{f}_o)|_t = (1/m_\Sigma)\,[\,\bar{E} - \int_{s.t} \bar{a}_t\, dm - \int_p \bar{a}_p\, dm\,] \tag{23}$$

where $m_\Sigma = m_s + m_t + m_p$; $\bar{E} = \int_{s.t.p} \bar{e}\, dm$

and m_s, m_t, m_p are the mass of the subsatellite, tether and shell, respectively. The subscript s.t or p designates the integration over the subsatellite and tether or the shell, respectively.

To obtain the Rayleigh-Ritz solution, u, v, w can be expanded in series form in terms of a set of admissible functions[8]

$$u = \sum_n \psi_n(x)A_n(t); \quad v = \sum_n \phi_n(x)B_n(t)$$

$$w = \sum_n \phi_n(x)C_n(t); \quad L(\overline{q}) = [L(u), L(v), L(w)]^T \tag{24}$$

Introduce

$$I_{\phi_m} = \int_{s.t} \phi_m dm \quad ; \quad I_{\psi_m} = \int_{s.t} \psi_m dm \; ; \; I_x = \int_{s.t} x \, dm$$

$$H_{\phi_n\phi_m} = \int_{s.t} \phi_n\phi_m dm; \; H_{x\phi_m} = \int_{s.t} x\phi_m dm; \; H_{\phi_n\psi_m} = \int_{s.t} \psi_m\phi_n dm$$

$$H_{xx} = \int_{s.t} x^2 dm \quad ; \; H_{x\psi_m} = \int_{s.t} x\psi_m dm \; ; \; H_{\psi_n\psi_m} = \int_{s.t} \psi_n\psi_m dm \tag{25}$$

Hence

$$\int_{s.t} u dm = \sum_m I_{\psi_m} A_m \quad ; \quad \int_{s.t} u' dm = \sum_m I_{\psi_m} A'_m$$

$$\int_{s.t} u'' dm = \sum_m I_{\psi_m} A''_m \; ; \; \int_{s.t} v \, dm = \sum_m I_{\phi_m} B_m \text{ , etc} \tag{26}$$

Mathematical expressions for the natural frequencies and mode shapes of the transverse vibrations of a shallow spherical shell with a completely free edge have been obtained in Ref.[9].

$$u_p = \sum_n A_{p_n}(t)\phi_{p_n} \tag{27}$$

where ϕ_{p_n} is the nth mode shape function

$$\phi_{p_n} = A_{jk}\left[\frac{a^{k+4}}{RD\lambda^4_{jk}} C_{jk}\zeta^k + J_k(\lambda_{jk}\zeta) + D_{jk}I_k(\lambda_{jk}\zeta)\right]\cos k(\beta+\beta_0) \tag{28}$$

a is the base radius of the shell

Since $\int_p \phi_{p_n} dm = 0$, hence

$$\int_p u_p \, dm = \int_p u'_p \, dm = \int_p u''_p \, dm = 0 \tag{29}$$

It is expected that the natural frequencies and shape functions will be modified by the presence of the tether system. However, in this paper we still adopt ϕ_{p_n} .

as the assumed shape functions, since the tether system mass is much less than that of the shell.

Equations (23) , (18) , (22) , (26) , (29) are substituted into equation (15) and the resulting vector equation is projected along the X, Y, Z axes. After integrating the projection along the X axis over the subsatellite and the tether the translational equation for the tether motion can be obtained

$$F_{tx} = \omega_c^2\{m^*_{st}(\ell''-u''_{p_0}+3u_{p_0})+\sum_m I^*_{\psi_m}(A''_m-3A_m) - 2\sum_m I^*_{\phi_m} C'_m+(2\alpha'+2\theta'-3)I^*_x$$
$$+m^*_{st}[(3-2\theta')h_x+(\phi''+2\psi'-4\phi+\gamma)h_y+(3\theta-\theta'')h_z]\}-\frac{m_p}{m_\Sigma}E_{stx}+\frac{m_{st}}{m_\Sigma}E_{px} \qquad (30)$$

where $m_{st} = m_s+m_t$; $m^*_{st} = m_{st}m_p/m_\Sigma$; $I^*_x = m_p I_x/m_\Sigma$

$$I^*_{\psi_m} = m_p I_{\psi_m}/m_\Sigma \; ; \; I^*_{\phi_m} = m_p I_{\phi_m}/m_\Sigma \qquad (31)$$

E_{stx}, E_{px} are components of $\bar{E}_{st}$, $\bar{E}_p$ along the X axis

$$\bar{E}_{st} = \int_{s.t} \bar{e}\, dm \; ; \; \bar{E}_p = \int_p \bar{c}\, dm \qquad (32)$$

and F_{tx} is the tether tension

The equations for the rotational motion of the tethered subsatellite can be obtained by the following operation

$$\int_{s.t} \bar{r}_t \times \text{equation (15)} = 0 \qquad (33)$$

By projecting equation (33) along the Y and Z axes, respectively, the rotational equations for the pitch(in-plane swing) and roll (out-of-plane swing) motions are obtained as

$$-\sum_m H^*_{x\phi_m}C''_m-3\sum_m(H^*_{x\phi_m}-h_x I^*_{\phi_m})C_m-2I^*_x(\ell'-u'_{p_0})-2\sum_m H^*_{x\psi_m}A'_m+H^*_{xx}\alpha''$$
$$+(\theta''+3\alpha+3\theta)(H^*_{xx}-h_x I^*_x)-I^*_x[(2\phi'+\gamma-\psi'')h_y-2\theta' h_z] = L_{ey}/\omega_c^2 \qquad (34)$$

$$\sum_m H^*_{x\phi_m}(B''_m+B_m)+3\sum_m(H^*_{x\phi_m}-h_x I^*_{\phi_m})B_m-H^*_{xx}(\gamma''+\gamma)-$$
$$(H^*_{xx}-h_x I^*_x)(\phi''+4\phi-3\gamma)-(I^*_x-\sum_m I^*_{\psi_m}A_m)h_y-I^*_x(\psi''+\psi)h_z = L_{ez}/\omega_c^2 \qquad (35)$$

where
$$H^*_{x\phi_m} = H_{x\phi_m} - I_x I_{\phi_m}/m_\Sigma$$
$$H^*_{x\psi_m} = H_{x\psi_m} - I_x I_{\psi_m}/m_\Sigma \qquad H^*_{xx} = H_{xx} - I_x^2/m_\Sigma \tag{36}$$

L_{ey}, L_{ez} are components of the torque $\overline{L}_e$, produced by the external force

By the following operations,

$$\int_{s.t} \psi_n \,[\text{Eq. } (15)]_x = 0 \quad (37n); \qquad \int_{s.t} \phi_n \,[\text{Eq. } (15)]_y = 0 \quad (38n)$$

$$\int_{s.t} \phi_n \,[\text{Eq. } (15)]_z = 0 \quad (39n)$$

the nth longitudinal and vibrational mode equations are obtained as

$$I^*_{\psi_n}(\ell''-u''_{p_0}+3u_{p_0}) + \sum_m H^*_{\psi_n\psi_m}(A''_m-3A_m) - 2\sum_m H^*_{\psi_n\psi_m}C'_m + (2\alpha'+2\theta'-3)H^*_{x\psi_n}$$
$$+I^*_{\psi_n}[(3-2\theta')h_x + (\phi''+2\psi'-4\phi+\gamma)h_y + (\theta''-3\theta)h_z] + \sum_m K_{mn}A_m = H^{(n)}_{ex} \tag{40n}$$

$$\sum_m H^*_{\phi_n\phi_m}(B''_m+B_m) + (\gamma''+4\gamma-\phi''-4\phi)H^*_{x\phi_n} +$$
$$I^*_{\phi_n}[(\phi''+4\phi-3\gamma)h_x + h_y - (\psi''+\psi)h_z] + H_{\phi_n\phi_n}\omega_n^2 B_n = H^{(n)}_{ey} \tag{41n}$$

$$\sum_m H^*_{\phi_n\phi_m}C''_m + 2I^*_{\phi_n}(\ell'-u'_{p_0}) + 2\sum H^*_{\phi_n\psi_m}A'_m - H^*_{x\phi_n}(\alpha''+\theta''+3\alpha+3\theta)$$
$$+I^*_{\phi_n}[(\theta''+3\alpha+3\theta)h_x + (2\phi'+\psi-\psi'')h_y - 2\theta' h_z] + H_{\phi_n\phi_n}\omega_n^2 C_n = H^{(n)}_{ez} \tag{42n}$$

where
$$H^*_{\psi_n\psi_m} = H_{\psi_n\psi_m} - I_{\psi_n\psi_m}/m_\Sigma \; ; \; H^*_{\psi_n\phi_m} = H_{\psi_n\phi_m} - I_{\psi_n}I_{\phi_m}/m_\Sigma$$
$$H^*_{\phi_n\phi_m} = H_{\phi_n\phi_m} - I_{\phi_n}I_{\phi_m}/m_\Sigma \tag{43}$$

and the terms H_{ex} . H_{ey}, H_{ez} results from the external force. $\sum_m K_{mn}A_m$, $\omega_n^2 B_n$, $\omega_n^2 C_n$ result from the elastic force[8]

The equations of shell motion

By the following operation

$$\int_p \overline{r}_o \times \text{equation } (6) = 0$$

the equations of the rotational motion of the shell can be obtained as (note: $\overline{c}$ in equation (6) includes the tether force acting on the shell which can be obtained by integrating equation (15) and it represents the effect of the motion of the tethered subsatellite system on the shell).

$$\psi''-\Omega_x^*\psi-(1+\Omega_x^*)\phi' = (1/J_x^*)\{m_{st}^*h_yh_z/\omega_c^2-h_y[I_x^*\alpha''+(I_x^*-h_xm_{st}^*)(\theta''+3\theta)$$

$$-2m_{st}^*(\ell'-u'_{p_0}-h_z\theta')-2\sum_m I_{\psi_m}^*A'_m-\sum_m I_{\phi_m}^*C''_m]+h_z[I_x^*(\gamma''+\gamma)$$

$$-(I_x^*-h_xm_{st}^*)(\phi''+4\phi)+\sum_m I_{\phi_m}^*(B''_m+B_m)]+(L_{Epx}+L_{epx})/\omega_c^2\} \tag{44}$$

$$\theta''-3\Omega_y^*\theta-2\sum_n I_1^{(n)}a\varepsilon'_{p_n}/J_y^* = (1/J_y^*)\{3h_z(m_{st}^*h_x-I_x^*)/\omega_c^2+h_z[m_{st}^*(\ell''-u''_{p_0}+3u_{p_0})$$

$$+\sum_m I_{\psi_m}^*(A''_m-3A_m)+2I_x^*(\alpha'+\theta')-2\sum_m I_{\psi_m}^*C'_m+h_ym_{st}^*(\phi''-4\phi+2\psi')]-h_x[\sum_m I_{\phi_m}^*C''_m$$

$$+2m_{st}^*(\ell'-u'_{p_0})+2\sum_m I_{\psi_m}^*A'_m-I_x\alpha''+h_ym_{st}^*(2\phi'+\psi-\psi'')]+(L_{Epy}+L_{epy})/\omega_c^2\} \tag{45}$$

$$\phi''+4\Omega_z^*\phi+(1-\Omega_z^*)\psi' = (-1/J_z^*)\{h_y(4m_{st}^*h_x-3I_x^*)/\omega_c^2+h_x[(\gamma''+\gamma)I_x^*-m_{st}^*(\psi''+\psi)h_z$$

$$+\sum_m I_{\phi_m}^*(B''_m+B_m)]-h_y[m_{st}^*(\ell''-u''_{p_0}+3u_{p_0})+2(I_x^*\alpha'+(I_x^*-m_{st}^*h_x)\theta'-\sum_m I_{\phi_m}^*C'_m)$$

$$+m_{st}^*(3\theta-\theta'')h_z+\sum_m I_{\psi_m}^*(A''_m-3A_m)]+(L_{Epz}+L_{epz})/\omega_c^2\} \tag{46}$$

where $\Omega_x^* = (J_z^*-J_y^*)/J_x^*$; $\Omega_y^* = (J_x^*-J_z^*)/J_y^*$; $\Omega_z^* = (J_y^*-J_x^*)/J_z^*$

$$J_x^* = J_x+m_{st}^*(h_y^2+h_z^2) \ ; \quad J_y^* = J_y+m_{st}^*(h_z^2+h_x^2)-I_x^*h_x$$

$$J_z^* = J_z+m_{st}^*(h_x^2+h_y^2)-I_x^*h_x \ ; \ I_1^{(n)} = \int_p x_p\phi_{p_n}dm \ ; \quad \varepsilon_{p_n} = A_{p_n}/a \tag{47}$$

$L_{\epsilon px}$, $L_{\epsilon py}$, $L_{\epsilon pz}$ are the components of torque, produced by $\overline{E}_{st}$ and $\overline{E}_o$, which appear in the tether force acting on the shell; L_{epx}, L_{epy}, L_{epz} are components of torque which are contributed by the external forces acting on the shell; and J_x, J_y, J_z are the principal moments of inertia of the undeformed shell.

By the following operation

$$\int_p \phi_{p_n} [\text{ equation } (6) \ ]_x = 0 \tag{48}$$

the nth shell elastic vibrational mode equation is obtained

$$\varepsilon''_{p_n}+(\Omega_n^2-3)\varepsilon_{p_n}+2I_1^{(n)}\theta'/M_na = [3I_1^{(n)}-F_{px}\rho_p^0+E_n/\omega_c^2]/M_na \tag{49n}$$

where E_n is the modal component of the external force, $\phi^o_{p_n}$ are ϕ_{p_n} at the point,O, M_n is the nth modal mass, $F_{..}$ is the component of the tether force acting on the shell along the X_* axis, $\Omega_n = \omega_{p_n}/\omega_c$ where ω_{p_n} is the natural frequency of the nth mode.

Equations (30) , (34) , (35) , (40n) , (41n) , (42n) , (44) , (45) , and (46), (49n) compose the complete system equations of motion.

Now, for our special case (Fig.3). O is along the shell yaw (i.e., X_*) axis, hence, $h_y = h_z = 0$ and it is assumed that there are no external forces acting on the system. By examination of the equations for this special case the following conclusions can be reached: (1) the shell roll, yaw motion, tether out-of-plane swing motion and elastic vibrations are decoupled from the shell pitch motion, shell elastic vibration, tether in-plane swing motion and in-plane elastic vibrations; (2) the shell pitch and elastic motions are coupled directly to each other through their rates; (3) since $I_1^{(n)} = 0$ for all shell elastic modes except for the axisymmetric modes, only the axisymmetric modes are coupled to the shell pitch motion; nonaxisymmetric modes are independent of the system motions, and would have to be controlled separately within the linear range.

2.3 Stability Analysis

It is well known that the shell pitch and roll-yaw motions are unstable about the present nominal orientation as $J_x > J_y$, $J_x > J_z$ without the attached tether system. In the present paper stability conditions for the tethered shell system will be developed when only tether flexibility is considered (the shell is considered rigid). In general, a finite number of elastic modes in the model is to be retained for practical purposes (truncated model). In the present paper a few such truncated modes are considered.

Rigid, constant length tether for in-plane motion

In this case all of the tether elastic modes are neglected and the tether length is fixed (without tension control). Hence, according to equations (34), (45) for our special case the equations of in-plane motion are simplified as follows:

$$K_1\alpha''+\theta''+3\alpha+3\theta = 0 \; ; \quad K_2\alpha''+\theta''-3\Omega_y^*\theta = 0 \tag{50}$$

where
$$K_1 = H_{xx}^*/(H_{xx}^*-I_x^*h_x); \; K_2 = -I_x^*h_x/J_y^* \tag{51}$$

$$\Omega_y^* = (J_x-J_z-m_{st}^*h_x^2+I_x^*h_x)/J_y^* \tag{52}$$

The system characteristic equation is given by

$$(K_1-K_2)\lambda^4+3(1-K_1\Omega_y^*-K_2)\lambda^2-9\Omega_y^* = 0 \tag{53}$$

since
$$K_1-K_2 = [H_{xx}^*J_y+h_x^2m_tm_p(m_s+m_t/4)/3m_t]/(H_{xx}^*-h_xI_x^*)J_y^* \tag{54}$$

if $h_x < 0$ then $K_1 - K_2 > 0$

The neutral stability conditions are

$$\Omega_y^* < 0 \quad (55); \qquad 1-K_1\Omega_y^*-K_2 > 0 \tag{56}$$

and
$$9(1-K_1\Omega_y^*-K_2)^2 + 4(K_1-K_2)9\Omega_y^* > 0 \tag{57}$$

It can be proved that if condition (55) is satisfied, then (56),(57) are also satisfied.

Meanwhile, if $h_x > 0$ the neutral stability conditions are

$$K_1 - K_2 > 0 \quad \text{and} \quad \Omega_y^* < 0 \tag{58}$$

but (58) is almost impossible to satisfy if $h_x > 0$; also according to (52) and (55) it is better to choose $h_x < 0$, so that the neutral stability conditions for in-plane motion are

$$h_x < 0 \quad \text{or} \quad h = -h_x > 0$$

$$J_z - J_x + m_{st}^* h^2 + I_x^* h = J_z - J_x + [(m_s + m_t)h^2 + (m_s + m_t/2)h\ ]\, m_p/m > 0 \tag{59}$$

Rigid, constant length tether for out-of-plane motion

According to equations (35) , (46) and (44) the equations for the rigid, constant length tether for out-of-plane motion are simplified as follows:

$$K_1\gamma'' - \phi'' + (3+K_1)\gamma - 4\phi = 0$$

$$K_3\gamma'' - \phi'' + K_3\gamma - 4\Omega_z^*\phi - (1-\Omega_z^*)\psi' = 0$$

$$\psi'' - \Omega_x^*\psi - (1+\Omega_x^*)\phi' = 0 \tag{60}$$

where $\quad K_3 = -I_x^* h_x / J_z^* \qquad \Omega_z^* = J_y - J_x + (m_{st}^* h_x^2 - I_x^* h_x)/J_z^*$

$$\Omega_x^* = (J_z^* - J_y^*)/J_x^* = (J_z - J_y)/J_x = \Omega_x \tag{61}$$

The system characteristic equation is given by

$$a_0\lambda^6 + a_2\lambda^4 + a_4\lambda^2 + a_6 = 0 \tag{62}$$

where $\quad a_0 = K_1 - K_3$

$$a_2 = -\Omega_x(K_1 - K_3) + 3 + K_1 + 4K_1\Omega_z^* - 5K_3 + K_1(1-\Omega_z^*)(1+\Omega_x)$$

$$a_4 = -\Omega_x(3 + K_1 + 4\Omega_z^* K_1 - 5K_3) + 4\Omega_z^*(3+K_1) - 4K_3 + (3+K_1)(1-\Omega_z^*)(1+\Omega_x)$$

$$a_6 = -4\Omega_x[(3+K_1)\Omega_z^* - K_3] \tag{63}$$

the neutral stability condition for this system is

$$\lambda^2 < 0 \tag{64}$$

i.e., λ^2 is a negative real number.

Since $a_0 > 0$, condition (64) is equated to the following conditions

$$a_2 > 0 \quad (65) \quad ; \quad a_2 a_4 - a_0 a_6 > 0 \quad (66) \quad ; \quad a_6 > 0 \quad (67)$$

and $$\Delta = a_2^2 a_4^2 + 18 a_0 a_2 a_4 a_6 - 4 a_0 a_4^3 - 27 a_0^2 a_6^2 - 4 a_6 a_2^3 > 0 \tag{68}$$

The conditions (65)-(67) are the Routh-Hurwitz conditions resulting from the cubic equation in the variable λ^2 [equation (62)] and condition (68) is from the condition that this cubic equation has three real roots.

It is known[2] that one of the necessary and sufficient stability conditions for out-of-plane motion of the shell system without the attached tether system is

$$\Omega_x = (J_z - J_y)/J_x < 0 \tag{69}$$

Now we assume condition (69) is still satisfied, so from condition (67) it is obtained that

$$\Omega_z^* > K_3/(3+K_1) \tag{70}$$

If the condition (70) is satisfied, it can be proven that the conditions (65) and (66) are also satisfied and it is demonstrated numerically that condition (68) is also satisfied for a variety of system parameters.

Hence, the neutral stability conditions for out-of-plane motion are

$$\Omega_x < 0 \quad (71) \; ; \qquad \Omega_z^* > K_3/(K_1+3) \tag{72}$$

The neutral stability conditions for a rigid, constant length tether are conditions (71) , and (72) together with condition (59)

Typical stability regions for in-plane and out-of-plane motion in the parameter space m_t, h are shown in Fig.5; it is seen that the stability region for in-plane motion is larger than that for the out-of-plane motion.

It can be proven that the out-of-plane motion will be asymptotically stable when damping of the shell roll angle is provided together with the conditions (71) and (72) .

Rigid variable length tether with Rupp's tension control law for in-plane motion

According to equations (30) , (34) , (45) the equations of motion with tension control are simplified as

$$\varepsilon''+2K\alpha'+(K+\beta_x)2\theta'-3K\varepsilon = F_{tx}/\omega_c^2 m^*_{st}\ell +3(K+\beta_x) = \Delta f$$

$$K_1\alpha''+\theta''+3\alpha+3\theta-K_4\varepsilon' = 0$$

$$K_2\alpha''+\theta''-3\Omega^*_y\theta-K_5\varepsilon' = 0 \tag{73}$$

where $\beta_x = h/\ell;\ \varepsilon = (\ell/\ell_c)-1\ ;\quad K= (m_s+m_t/2)/(m_s+m_t)$

$$K_4= 2I^*_x\ell/(H^*_{xx}+hI^*_x)\quad ;\quad K_5 = 2m^*_{st}h\ell/J^*_y \tag{74}$$

For Rupp's tension control law[10]

$$\Delta f = -(K_\varepsilon\varepsilon+K_{\varepsilon'}\varepsilon') \tag{75}$$

the system characteristic equation is developed and the Routh-Hurwitz criteria applied. After some complicated algebraic manipulations the following expressions for the principal minors are obtained

$$D_1 = (K_1-K_2)K_{\varepsilon'}$$

$$D_2 =(K_1-K_2)K_{\varepsilon'}[(K_1-K_2)K^*_\varepsilon+2K(K_5K_1-K_4K_2+K_4-K_5)+2\beta_x(K_5K_1-K_4K_2)]$$

$$D_3=(K_1-K_2)K^2_{\varepsilon'}[-3\Omega^*_y(K_5K_1-K_4K_2)(2K_1-KK_4)\beta_x+6K(K_5K_1-K_4K_2+K_4-K_5)^2/K_4]$$

$$D_4 =9(K_1-K_2)K^2_{\varepsilon'}\{(A^*\Omega^{*2}_y+B^*\Omega^*_y+C^*)K^*_\varepsilon +(8K(1-K_1)/K_5)[(K_4K_2-K_5K_1)\Omega^*_y$$
$$-(K_5/K_4)(K_5K_1-K_4K_2+K_4-K_5)]^2\}$$

$$D_5 = -648\Omega^*_yK^3_{\varepsilon'}K(1-K_1)[(K_4K_2-K_5K_1)\Omega^*_y -K_5(K_5K_1-K_4K_2+K_4-K_5)/K_4]^2/K_5$$

$$D_6 = -9\Omega^*_yK^*_\varepsilon D_5 \tag{76}$$

where $K^*_\varepsilon= K_\varepsilon-3K$

$$A^* =4H^*_{xx}m_pm_t^2(m_s+m_t/4)^2h^2\ell^4/[9m_t(m_s+m_t)(H^*_{xx}+hI^*_x)^2J^{*2}_y$$

$$B^* = 8Kh^2\ell^3 m_p m_t (m_s+m_t/4) J_y I_x^* / [3m_\Sigma (H_{xx}^*+hI_x^*)^2 J_y^{*2}]$$

$$C^* = 4I_x^*\ell K(J_y+m_{st}^*h^2)J_y^2/[(H_{xx}^*+hI_x^*)J_y^{*3}] \tag{77}$$

The necessary and sufficient stability conditions are

$$D_i > 0 \quad (i=1, \cdots\cdots 6) \tag{78}$$

Since h > 0 (or h_x < 0) according to equations (54) and (51)

$$K_1 - K_2 > 0, \qquad K_1 < 1$$

Hence, from $D_1 > 0$, $D_5 > 0$, and $D_6 > 0$, there results,

$$K_{\varepsilon'} > 0, \; K_\varepsilon^* > 0 \text{ and } \Omega_y^* < 0 \tag{79}$$

Meanwhile, since

$$K_5K_1-K_4K_2+K_4-K_5 = 2I_x^*\ell J_y/[(H_{xx}^*+hI_x^*)J_y^{*}] > 0$$

$$K_5K_1-K_4K_2 = 2h\ell^3 m_p\, m_t(m_s+m_t/4)/\,[3(H_{xx}^*+hI_x^*)J_y^* m_\Sigma] > 0 \tag{80}$$

$$2K_1-KK_4 = 2\ell^2 m_t(m_s+m_t/4)/[3(H_{xx}^*+hI_x^*)m_{st}] > 0$$

and $\quad B^{*2} - 4A^*C^* < 0$

thus, if $\quad K_{\varepsilon'} > 0,\; K_\varepsilon^* > 0,\; \Omega_y^* < 0,$

then $\quad D_i > 0 \quad (i= 1, \cdots\cdots 6)$

So the necessary and sufficient conditions for in-plane motion stability with Rupp's tension control law are:

$$\Omega_y^* < 0; \; K_{\varepsilon'} > 0 \text{ and } K_\varepsilon^* > 0 \tag{81}$$

Stability conditions for flexible tether

When tether flexibility is considered it is difficult to get analytical results for the stability conditions because of the high order of the system. However, it is found numerically that the system stability conditions are unchanged both for in-plane motion (with constant length tether or with Rupp's tension control law) and out-of-plane motion (with constant length tether) when the tether flexible modes are included in the system model, for a variety of system parameters (for example, for variations of h, m_s, and ℓ).

2.4 Optimal Tension Control Law for In-Plane Motion during Stationkeeping

In the last section it has been demonstrated that the system is asymptotically stable with Rupp's tension control law for in-plane motion. However, in order to improve the transient responses two alternate optimal control laws are introduced. One is an optimal control law based on tether length, in-plane swing angle and shell pitch angle and their rates for the rigid massive tether model; another is an optimal control law, which includes additional feedback of the tether vibrational modes and their rates. For the system with the state variable format equations the optimal control, U, which minimizes the performance index

$$J = \int_0^\infty (X^TQX + U^TRU) \, dt$$

is given by $U = - (R^{-1}B^TP) = -KX$

where X is the state variable

Q is a positive semi-definite state penalty matrix

R is a positive definite control penalty matrix and

P is the positive definite solution to the steady state Riccati matrix equation

$$-PA - A^TP + PBR^{-1}B^TP - Q = 0$$

The optimal control law, based on the tether length, in-plane swing angle, and shell pitch angle and their rates for the rigid tether model (with state variables

$X^T = [\quad \alpha, \theta, \varepsilon, \alpha', \theta', \varepsilon' \quad])$ takes the form:

$$\Delta f = -(K_\alpha \alpha + K_\theta \theta + K_\varepsilon \varepsilon + K_{\alpha'} \alpha' + K_{\theta'} \theta' + K_{\varepsilon'} \varepsilon')$$

For the system parameters: $m_s = 10000$kg, $m_t = 500$kg, $= 1$km, $m_t = 8.35$kg, $h = 0.08$km some typical system simulation (three vibrational modes are included) results show that the transient responses for the rigid tether optimal control law is better than that for Rupp's tension control law for some of the control gains, especially for the damping of the tether and shell pitch angles. But it is also found that the system could be unstable for some of control gains (Table 1).

It is obvious that the transient responses, when the optimal control law includes feedback of the vibrational modes, are better than the responses based on the previous control law. The improvement is especially noted in the damping of the tether vibrational modes. A typical comparison of transient responses for the three different tension control laws is shown in Figs. 6a, 6b, and 6c.

2.5 Conclusions

The orbiting shallow spherical shell pitch and roll-yaw motion are unstable when the symmetry axis nominally follows the local vertical. However, it is suggested that gravitational stabilization could be achieved by including a tethered subsatellite system to provide the favorable moment of inertia distribution. The tether could be connected at the end of a rigid boom which is attached to the shell's apex. The equations of motion for such a tethered shallow spherical shell in orbit with the present nominal orientation have been developed in this paper.

The shell roll-yaw motion, tether out-of-plane swing motion, and the tether out-of-plane elastic vibrations are decoupled from the shell pitch, shell elastic vibration, tether in-plane swing motion and tether in-plane elastic vibrations. For given shell and tethered subsatellite system parameters a suitable rigid boom length could be chosen in order to provide a gravitational stable structure both for in-plane and out-of-plane motion. The in-plane motion of the system could be asymptotically stable with Rupp's tension control law. It is demonstrated numerically that the flexibility of the tether would not affect the stability conditions for the constant length tether or for the variable length tether with Rupp's tension control law for a variety of system parameters. The transient responses can be improved significantly, especially for the damping of the tether and shell pitch motion, by an optimal control law for the variable length tether model. It is also seen that the system could be unstable when the effect of tether flexibility is included if the control gains are not chosen carefully. The transient responses can be further improved by including the state feedback of the tether vibrational modes into the optimal control law, especially for the damping of the tether vibrations.

Extensions to the present paper could consider the effect of the shell flexibility on the system stability and control and some kind of active control could be introduced (in addition to tether tension control) to improve system performance. Additional control will be required to provide for out-of-plane damping of rigid motions and vibration suppression.

Table 1. Stability Characteristics for Different Control Gains

R	K_α	K_θ	K_ε	$K_{\alpha'}$	$K_{\theta'}$	$K_{\varepsilon'}$	Stability
1	6.938	5.875	6.114	2.722	3.912	4.917	unstable
2	5.456	4.841	6.034	1.637	2.455	4.377	stable
5	4.296	4.059	5.984	0.583	1.058	3.811	stable
10	3.812	3.742	5.967	0.012	0.304	3.478	stable
20	3.536	3.565	5.959	-0.405	-0.249	3.217	stable
30	3.437	3.503	5.956	-0.591	-0.499	3.095	unstable

where $Q = \delta_{ij}$

Table 2. Control Gains

	K_α	K_θ	K_{η_1}	K_{η_2}	K_ε	$K_{\alpha'}$	$K_{\theta'}$	$K_{\eta'_1}$	$K_{\eta'_2}$	$K_{\varepsilon'}$
Fig. 6a.	0	0	0	0	6	0	0	0	0	3.46
Fig. 6b.	5.46	4.84	0	0	6.03	1.64	2.46	0	0	4.38
Fig. 6c.	9.19	7.50	59.2	119	6.27	4.13	5.85	-0.22	-0.09	5.63

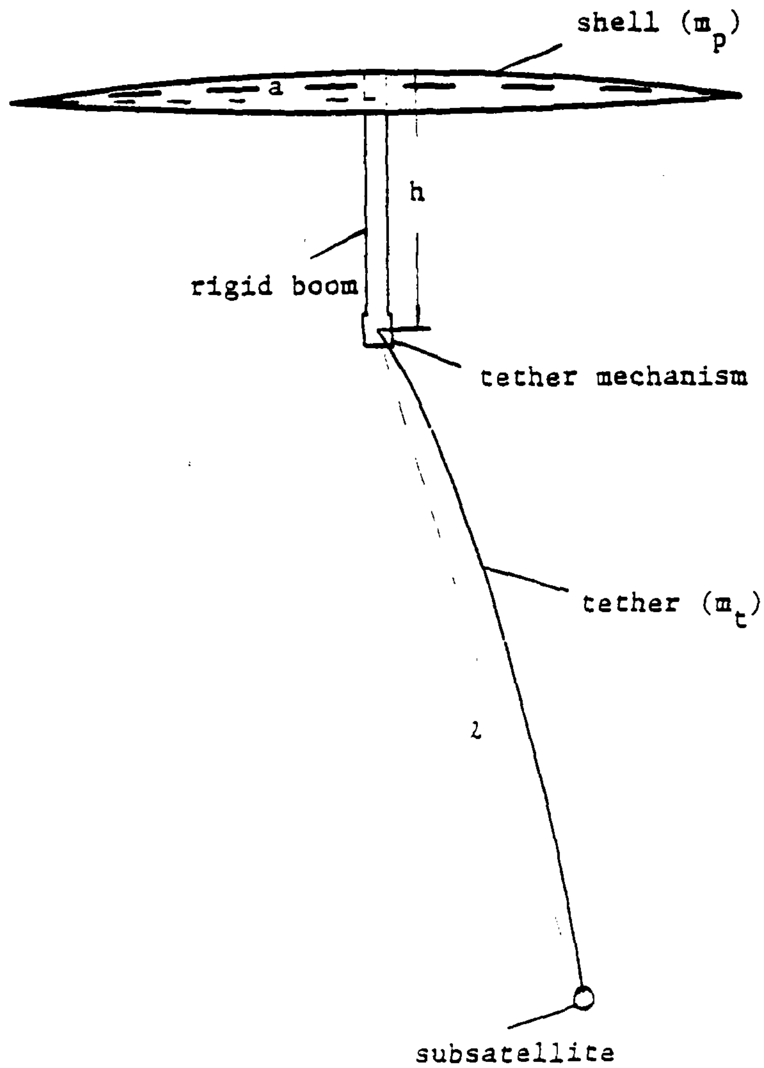


Fig. 3. Tethered Antenna/Reflector System

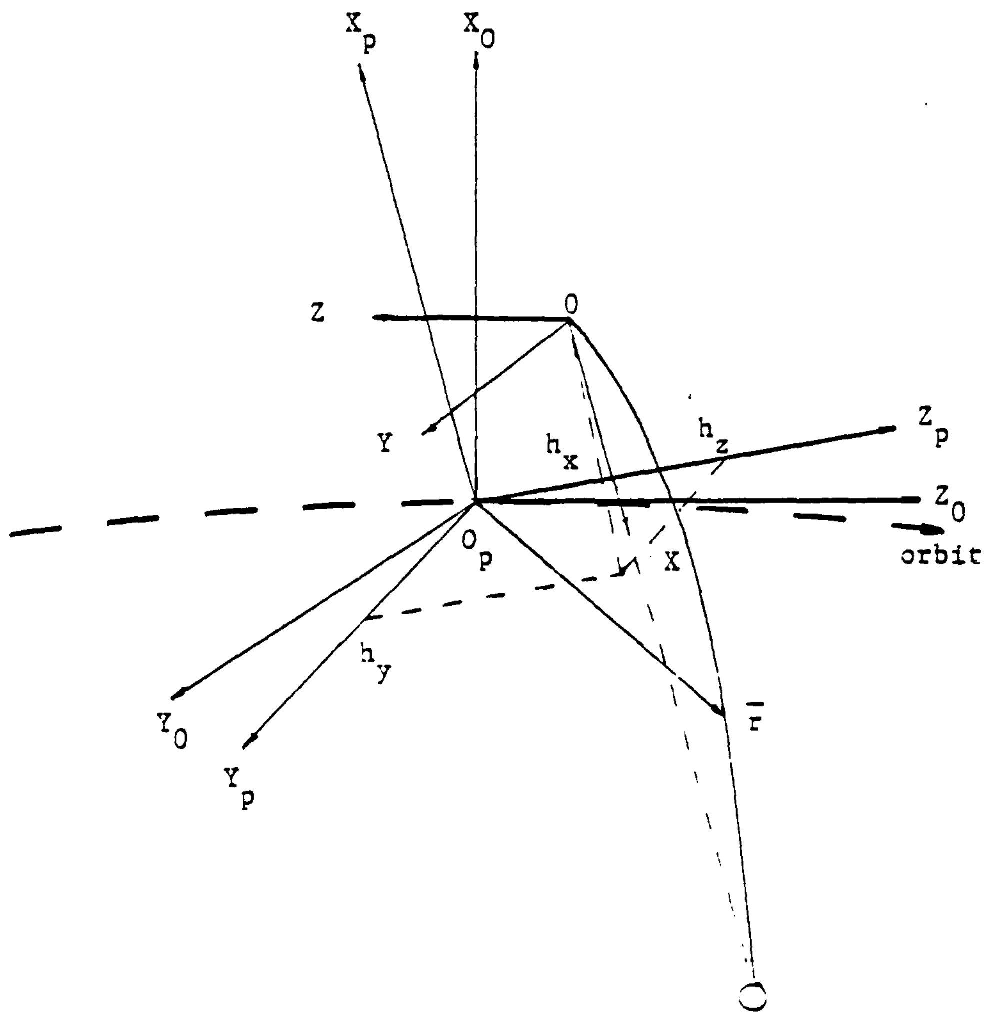


Fig. 4. Coordinate Systems

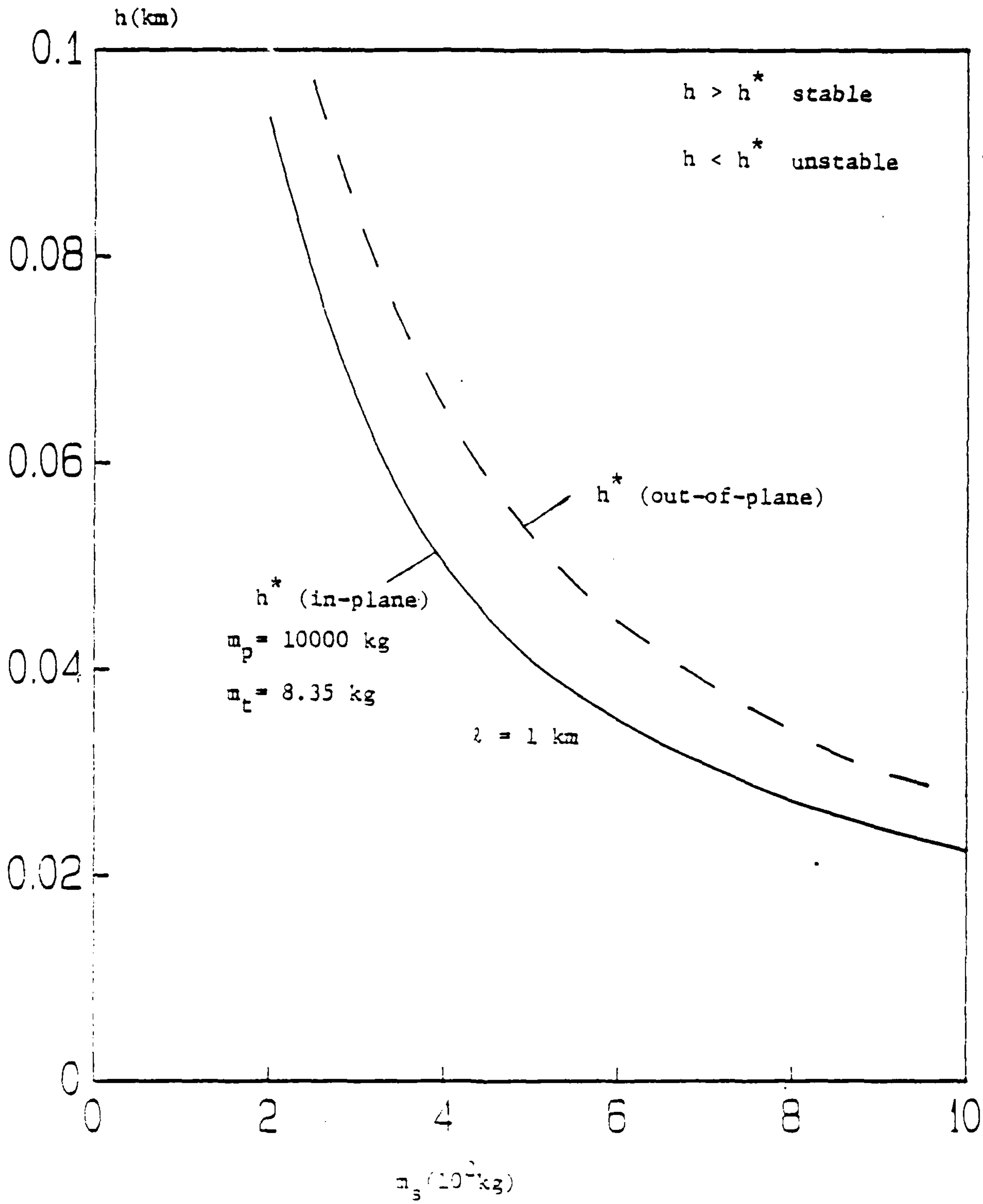


Fig. 5. Stability Regions for In-Plane and Out-of-Plane Motion

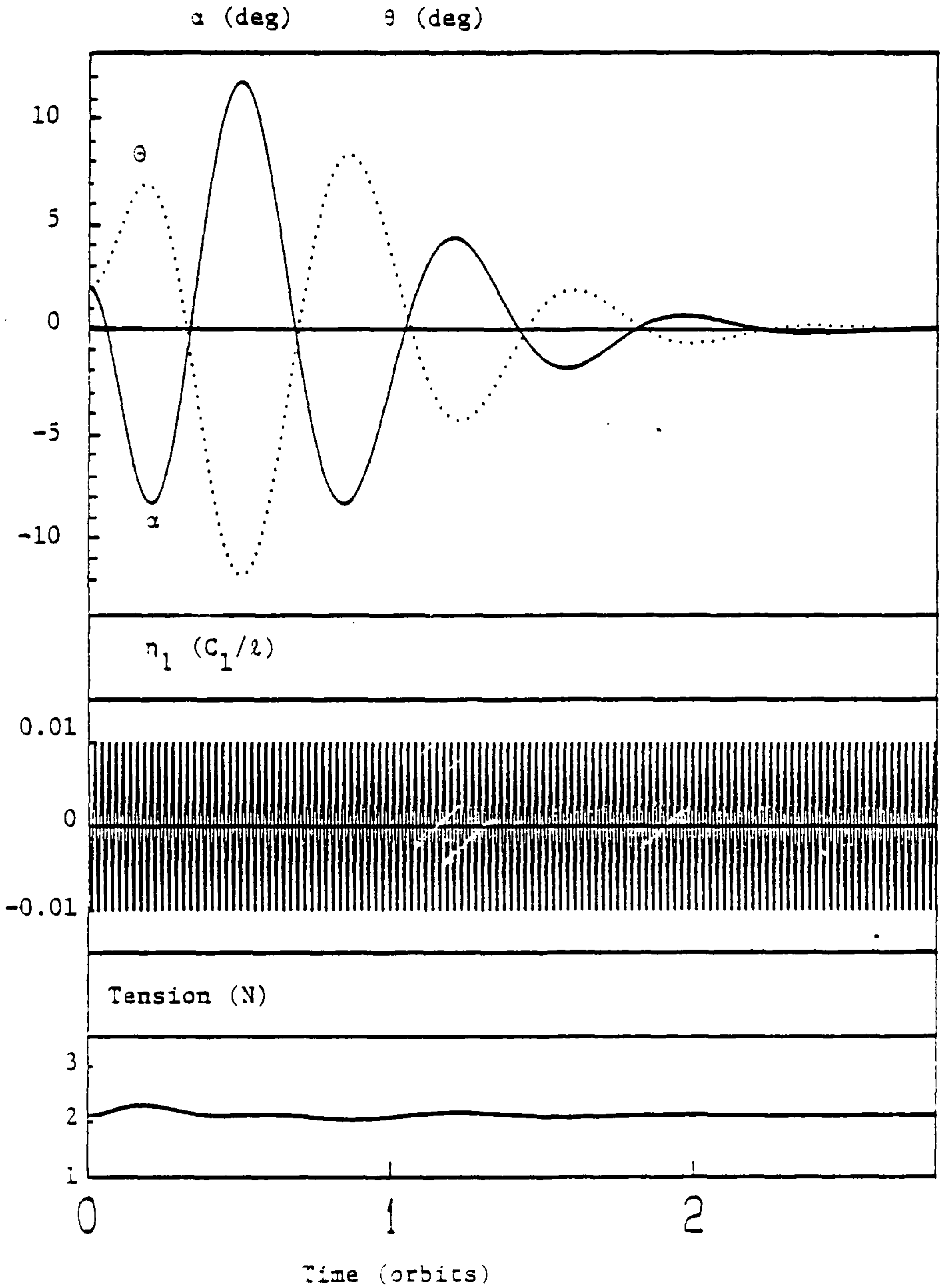


Fig. 6a. Transient Responses for Rupp's Control Law

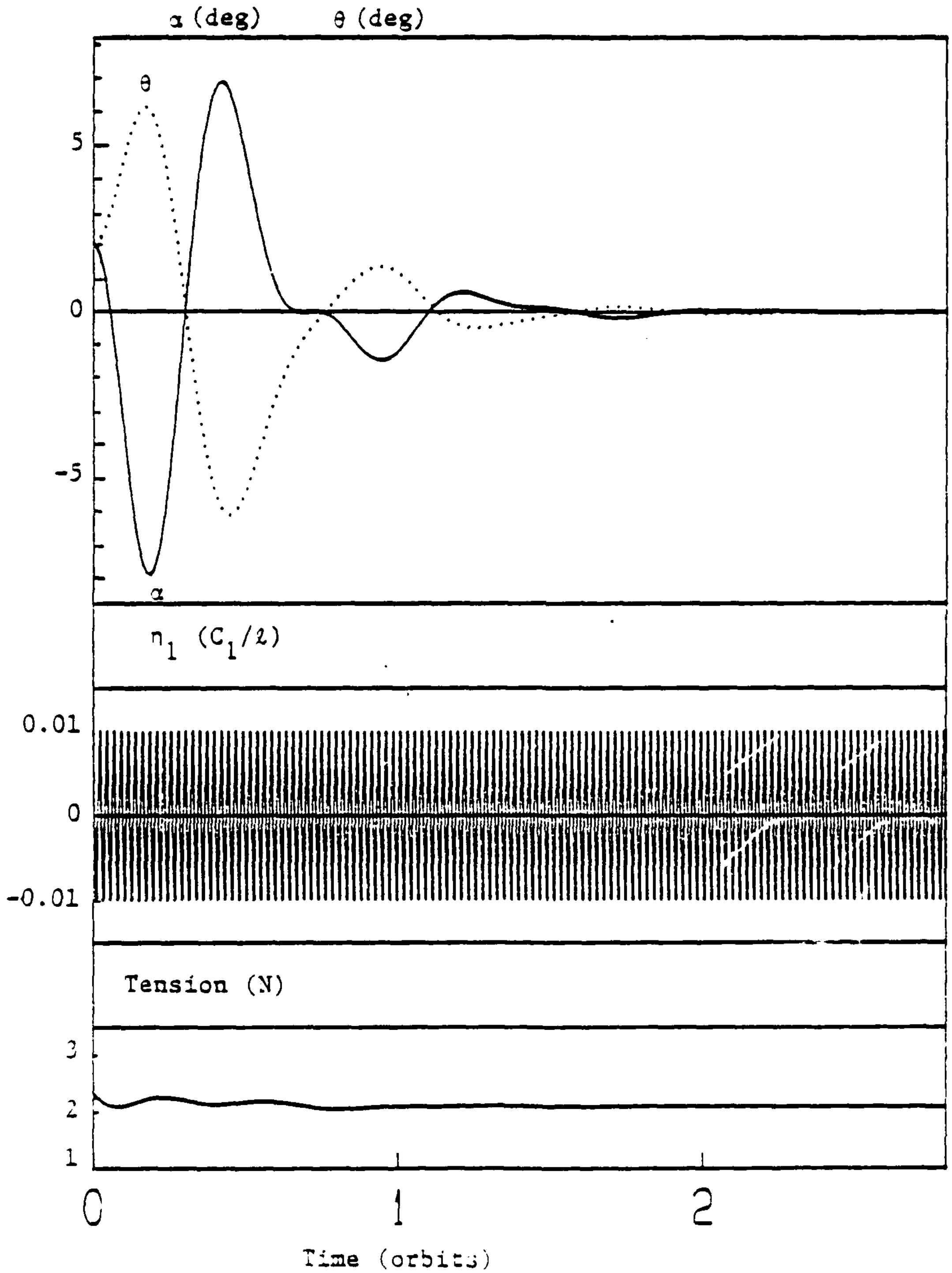


Fig. 6b. Transient Responses for Rigid Tether Model Optimal Control Law

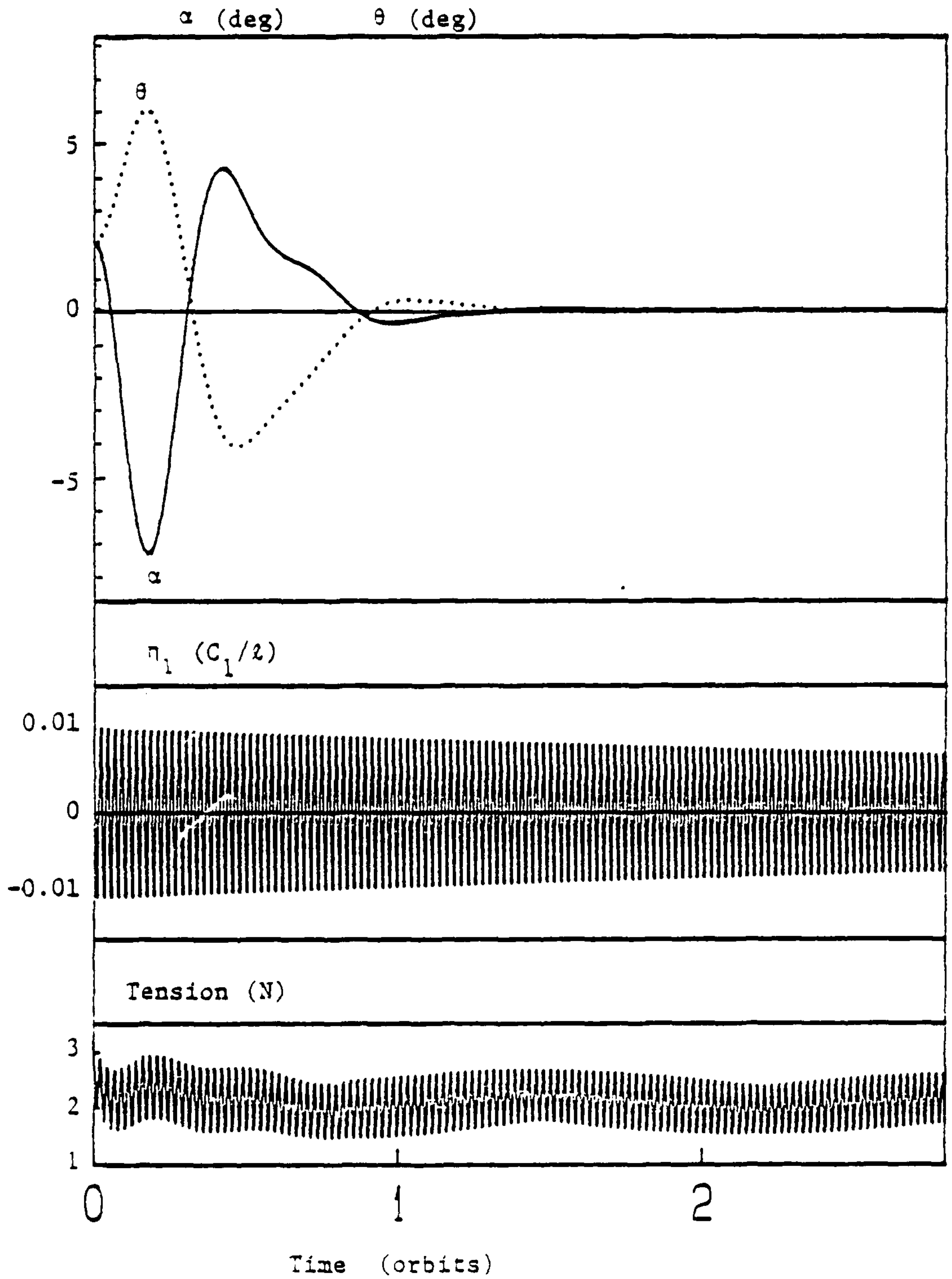


Fig. 6c. Transient Responses for Flexible Tether Model Optimal Control Law

3. REVIEW OF THE CONTROL OF TETHERED SATELLITE SYSTEMS

This chapter reviews the steps in the development of control laws for Shuttle/Platform-Tethered Subsatellite systems. The tethered subsatellite systems have been proposed for numerous applications. This has led to many investigations dealing with the dynamics and control of such systems during their deployment, station keeping, and retrieval. A brief comparison of control laws used by various investigators is described here in order to evaluate different control methods for tethered subsatellite systems.

Finally, recommendations are made as to the suitability of the different control laws for adaptation with the proposed orbiting tethered reflector systems.

3.1 Deployment

This operation involves moving the subsatellite from the Shuttle/Platform to a distance as much as 100km from the Shuttle/Platform. Since most of the useful mission activities start after deployment of the subsatellite, it is desirable to deploy the subsatellite to the operational altitude in as short a time as possible.

Deployment can be carried out either using a passive procedure as suggested by Kane and Levinson and described in Ref. [11] or with the help of active control similar to that proposed by Rupp [10]. The passive procedure is initiated by placing the tether (except its end point) outside the Shuttle and allowing it to float freely. Next, the payload mass, m_s, attached to one end of the tether is ejected from the Shuttle and performs a free flight until the tether becomes taut at which time the payload is subjected to impulse affecting the motion. Impacts and free flight occur alternately until enough energy has been dissipated during impacts so that no further ones occur. The tether then becomes permanently taut and the system behaves like a spherical pendulum. If some viscous damping is provided, the pendulum settles along the local vertical. Whether deployment in the desired direction (upward or downward) occurs or not, the duration of deployment depends on the initial ejection velocity.

Deployment is more likely to be carried out using an active control system to guarantee adequate dynamic performance. Rupp [10] first made a preliminary treatment of the dynamics of the Shuttle-Tethered-Subsatellite system in which the motion is assumed to occur only within the orbital plane and tether mass is neglected, and set up a tension control law in the form

$$T = K_1L + C_1\dot{L} + K_2L_c \qquad (82)$$

where L and $\dot{L}$ are instantaneous length and length rate, respectively. L_c is a commanded length while K_1, K_2 and C_1 are a set of constants. L_c is changed in steps until the final tether length is attained. Implementation of the control law requires the measurement of tether tension, rate of the tether deployment and instantaneous tether length. The latter two can be measured by a pulley kept in frictional contact with the tether. The tension can be measured by a spring damper arrangement on the same pulley. The measurements are fed into a computer which calculates the required torque that must be produced by the motor driving the tether reel system.

In the dynamic simulation using Eq. (11) Rupp [10] used $K_2 = K_1 - 3m_s\omega^2$ while two values of K_1 ($5m_s\omega^2$ and $7m_s\omega^2$) were chosen. [ω is the assumed (circular) orbital frequency of the Shuttle orbit.] The coefficient, C_1, was used to critically damp the longitudinal stretching oscillations. The control law was quite effective in damping in-plane motion during deployment, but, for out-of-plane motion was not considered.

Baker et al. [12] treated the three dimensional dynamics and control including the inertial effect of the tether mass, aerodynamic heating, and aerodynamic forces on the tether and subsatellite. The form of the control law used was the same as in Rupp [10] with the exception of modification of the commanded length for desired deployment and retrieval maneuvers. The modified tension law was in the form as follows:

$$\frac{T}{m_s + \frac{1}{2} m_t} = (R^2-3)\omega^2 L + 2\xi_c R\omega\dot{L} - R^2\omega L_c \qquad (83)$$

where m_s and m_t are the mass of subsatellite and tether, respectively. R is the ratio between the control law stretch frequency and orbital frequency, while ξ_c is the control law damping ratio. A commanded length of the form

$$L_c = K_1 L + K_3 \qquad (84)$$

was suggested.

The combinations of exponential and uniform rates of change of length can be expressed in the form

$$L_c = \begin{cases} L_i \exp(ct) & L_i \leq L_c < L_1 \\ L_i (1+ct) & L_i \leq L_c < L_2 \\ L_1 - L_2 - L_1 \exp(-ct) & L_2 \leq L_c \leq L_f \end{cases} \qquad (85)$$

where c is a positive constant, L_i and L_f are initial and final lengths, respectively, while L_1 and L_2 are two intermediate lengths.

Deployment is basically a stable operation [Baker et al.][12] ; however, towards the end of deployment, when aerodynamic effects become important bounded steady inplane rotational and elastic oscillation may result. In addition, out-of-plane rotations and vibrations occur for eccentric orbits inclined to the equatorial plane. This motion does not cause any serious problems and can be eliminated once the deployment is completed.

Kalaghan et al. [13] used a finite element approximation as a discretized mathematical model of the Shuttle-Tethered-Subsatellite system which includes tether mass effects; however the form of the tether control law was similar to that in Rupp [10] and Baker et al. [12]

Basically similar results were obtained when deployment is carried out using a length rate law [Misra and Modi][14] of the form.

$$L' = L_c' \, [1+\underline{K}^T\underline{X}] \tag{86}$$

where $\underline{K}$ and $\underline{X}$ are gains and state vectors, respectively.

Subsequently, Bainum and Kumar [15] first developed an optimal control law based on an application of the linear regulator problem, with control provided only by modulating the tension level as a function of the difference between the actual and commanded length ($\varepsilon = L_c/L - 1$), actual length rate (ε'), inplane tether line swing angle, α, and its rate α', i.e,

$$T = K_{\varepsilon}\varepsilon + K_{\varepsilon}\,\varepsilon' + K_{\alpha}\alpha' + K_{\alpha'}\alpha' + T_c \; ; \; T_c = 3\omega^2 L_c m_s \tag{87}$$

The system was idealized as two point masses connected by a massless, inextensible tether with the system moving in a nearly-circular orbit. The dynamic effects of orbital eccentricity and the earth's oblateness were neglected.

In the dynamic simulation using Eq. (87) Bainum and Kumar used a commanded length, L_c, exponentially increasing with time as in Kalaghan et al. [13], with suitable modification for the deployment , i.e.

$$L_c = 10 + 94990 \, (1-e^{-t/p}) \tag{88}$$

where p is a positive constant. With this commanded length used in conjunction with the optimal feedback gains, deployment requires a moderate duration of time and has better damping characteristics, while comparing this result with that in Kissel [Baker et al.][12] and SAO [Kalaghan et al.] [13]. Kissel's deployment simulation without aerodynamics and no out-of-plane motion required approximately 6.7 hr. to deploy the subsatellite to 100km, whereas the SAO deployment using Rupp's [10] law with atmospheric effects included required approximately 10 hr., to deploy the subsatellite to 100 km.

3.2 Stationkeeping

This task involves the maintenance of the subsatellite in the desired equilibrium configuration. For the present proposed aerodynamic test mission, the desired equilibrium configuration involves a subsatellite deployed 100 km below (or above) the Shuttle Orbiter (Shuttle Orbiter altitude ~ 220 km) along the local vertical. Under the influence of external disturbances, the subsatellite deviates from this desired position.

Bainum and Kumar [15] used optimal control law theory to investigate station keeping for the Shuttle Tethered Sub-satellite system. The system response to various initial conditions with all the external disturbing forces absent was studied. Typical results showed that the time constant for the least damped mode corresponding to the optimal gains (0.93 orbit) was shorter than those associated with Rupp's law (1.581 orbits, in-plane tuning). In Rupp's control law, the length and length rate gains were selected for in-plane and out-of-plane control based on the system natural frequencies of the in-plane and out-of-plane modes, respectively. In accordance with optimal control theory the feedback control gains were selected in order to minimize the performance index.

$$J = \int_0^{\infty} (X^T Q X + U^T R U)\, dt \qquad (89)$$

$$U = -(R^{-1}B^{T}p)X = -KX \qquad (90)$$

The system response with lower overshoots, shorter settling times, and with comparable power and tension levels was obtained by applying modern control theory. Maximum tensile acceleration for this case was 0.525 m/sec^2 for the optimal control and 0.505 m/sec^2 for Rupp's [10] in-plane tuning control.

When the effect of atmosphere on the tether was considered with Rupp's [10] control, the system response had less damping than with the optimal control. This reflected the greater stiffness in the optimally controlled system. Also, Rupp's control had a tendency to pull the subsatellite to a higher altitude, unlike the optimal control which had a tendency to deploy the subsatellite further into the atmosphere. It is evident that the control law based on the linear regulator theory resulted in a superior performance when compared to Rupp's control law in the station keeping mode.

Then the previous work was later extended to Platform - Tethered - Subsatellite systems by Bainum, Woodard, and Juang [4], where a 2-dimension mathematical model of open and closed loop in-orbit plane dynamics of a space Platform-Tethered-Subsatellite system was treated. The control was assumed to be provided by both modulation of the tether tension level and momentum type platform-mounted devices. The control laws for controlling tether tension and platform pitch angle, respectively, were,

$$T_L = K_{\varepsilon\varepsilon}\varepsilon + K_{\varepsilon\varepsilon'}\varepsilon' + K_{\varepsilon\alpha}\alpha_v + K_{\varepsilon\alpha'}\alpha'_v + K_{\varepsilon\theta}\theta_v + K_{\varepsilon\theta'}\theta'_v \quad (91)$$

$$T_\theta = K_{\theta\varepsilon}\varepsilon + K_{\theta\varepsilon'}\varepsilon' + K_{\theta\alpha}\alpha_v + K_{\theta\alpha'}\alpha'_v + K_{\theta\theta}\theta_v + K_{\theta\theta'}\theta'_v \quad (92)$$

where $\varepsilon = L/L_c - 1, \varepsilon', \alpha_v = \alpha - \alpha_{eq}, \alpha'_v, \theta_v = \theta - \theta_{eq}, \theta'_v$ and K's are state variables and gains, respectively. The angle, θ, describes the orientation of the platform with respect to the local vertical and the angle, α, represents the angular displacements of the tether line relative to a local normal in the platform, L_c is the nominal reference length, and the subscript "eg" refers to equilibrium values.

The numerical results showed that tether line swing motion was damped, requiring about 1.75 hr to reach the nominal value, whereas the platform pitch motion was damped out within approximately 1.0 hr when corresponding feedback gains and initial conditions were determined.

It was proved that: (1) within the linear range the system is controllable with momentum-type control on the platform and with tension modulation in the tether line; (2) the linear system is observable with tether length and length rate measurements only; (3) the tether attachment point offset increases the system's natural coupling and improves transient performance in the least damped mode, but at the cost of slightly larger control force amplitudes.

Continuing, Fan Ruying and Bainum [5] developed a 3-dimensional mathematical model of the open and closed loop dynamics of a Space Tethered-Platform-Subsatellite system and used the same form of the control law as in Ref. [15].

It was assumed that the control could be realized through appropriate modulation of the tension in the tether line and the momentum type controller for the platform pitch, roll and yaw rotation, i.e.

$$T_L = K_{\varepsilon\varepsilon}\varepsilon + K_{\varepsilon\theta}\theta_v + K_{\varepsilon\alpha}\alpha_v + K_{\varepsilon\varepsilon'}\varepsilon' + K_{\varepsilon\theta'}\theta'_v + K_{\varepsilon\alpha'}\alpha'_v \qquad (92)$$

$$T_\theta = K_{\theta\varepsilon}\varepsilon + K_{\theta\theta}\theta_v + K_{\theta\alpha}\alpha_v + K_{\theta\varepsilon'}\varepsilon' + K_{\theta\theta'}\theta'_v + K_{\theta\alpha'}\alpha'_v \qquad (93)$$

(in-plane)

$$T_\phi = K_{\phi\phi}\phi + K_{\phi\psi}\psi + K_{\phi\gamma}\gamma + K_{\phi\phi'}\phi' + K_{\phi\psi'}\psi' + K_{\phi\gamma'}\gamma' \qquad (94)$$

$$T_\psi = K_{\psi\phi}\phi + K_{\psi\psi}\psi + K_{\psi\gamma}\gamma + K_{\psi\phi'}\phi' + K_{\psi\psi'}\psi' + K_{\psi\gamma'}\gamma' \qquad (95)$$

(out-of-plane) .

where the angles θ, ϕ, ψ, and θ', ϕ', ψ' are platform pitch, roll and yaw angles and their rates, respectively, α, α', γ, γ' are tether line in-plane and out-of-plane swing angles and angular rates, respectively.

The numerical results showed that the platform pitch angle and tether line swing angle damped, both requiring about 1.5 hr for the initial conditions selected. For out-of-plane motion the platform roll and yaw angles, and the tether line swing angle damped requiring about 2.5 hr, 3 hr and 1.5 hr, respectively.

For the case where the tether attachment point offset was only along the roll axis, it was verified that both the in-plane and out-of-plane subsatellite systems are controllable when tension modulation on the tether and momentum type control are available. Both subsystems are observable if the length of the tether, and the platform rotation angles together with their rates are available.

The tether attachment point offset, which is the source of the system's natural coupling, is an important factor in establishing system controllability and observability. For the case of no attachment offset, the rotation of the platform will not affect the subsatellite out-of-plane swing; in other words, the effect of higher order terms should be considered, or other means of control, such as by placing an actuator on the subsatellite to control the tether line out-of-plane swing, should be augmented.

The investigation of the effect of tether flexibility on the (in-plane) stability regions as a function of the tether tension control parameters during the station keeping was further developed by Liu Liangdong and Bainum [8] where an alternate optimal control strategy which included additional feedback of the first vibrational mode and its rate was introduced. The formulation of tension level control was in the form

$$T = -(K_\varepsilon \varepsilon + K_{\varepsilon'}\varepsilon' + K_\alpha \alpha + K_{\alpha'}\alpha' + K_{\eta_1}\eta_1 + K_{\eta_1'}\eta_1') \qquad (96)$$

where η_1, η_1' is the first flexible modal amplitude (non-dimensional) state variable and its rate, respectively. The system is controllable using only tether tension control and also observable with the measurement of $\varepsilon, \varepsilon' \alpha, \alpha'$, or only $\varepsilon, \varepsilon'$; η_1, η_1' are available either through estimation or by direct measurement.

According to numerical comparison of the transient responses it was seen that Rupp's control law could be used to control the the in-plane swing angle successfully during station keeping but it was not very effective for damping the in-plane vibrations. The transient responses for Bainum and Kumar's [15] optimal law based on $\varepsilon, \varepsilon' \alpha, \alpha'$ were faster than those for Rupp's [10] control law. The further improvements in transient response in both the in-plane swing angle and the first vibration modal amplitude were made by including the state feedback of the first vibrational mode into the optimal control law. An impovement was also apparent in the damping of the second mode due to the coupling between the first and second modes.

3.3 Retrieval

The retrieval is basically an unstable procedure. Retrieval can be carried out by letting the commanded length, L_c, reduce with time. L_c can be decreased in steps [Rupp] or it can be an exponentially decreasing continuous function of time, such as

$$L_c = L_o e^{-t/p} \tag{97}$$

In either case, the rotational as well as vibrational motions are unstable in the absence of active control, because the negative damping introduced during retrieval is proportional to L/L and, in practice, the damping level required to guarantee stability is not always available.

Rupp's [10] and Baker's et al. [12] tether tension control laws were used for retrieval, but large amplitude out-of-plane-swing oscillations which approached 60° and

45°, respectively, resulted, depending on the initial conditions.

With Bainum and Kumar's [15] optimal control strategy for retrieval, the in-plane response is better than that obtained by Baker et al. [12], however, out-of-plane motion is hardly improved.

To restrict out-of-plane oscillations to reasonable bounds, nonlinear control strategies must be used. Xu et al. [16] showed that a satisfactory length rate law is of the form,

$$\frac{L'}{L} = K_\theta[1 + K_{\alpha'}\alpha' + K_{\gamma'}\gamma'^2] \qquad (98)$$

where $K_{\alpha'}$ and $K_{\gamma'}$ are negative constants K_θ is a negative function of the true anomaly θ, α is the in-plane tether swing angle, and γ the out-of-plane tether swing angle. Modi et al. [17] considered some nonlinear control strategies with a tension control law of the form,

$$T = K_L L + K_{L'}L' + K_{\gamma'}\gamma'^2 + T_o \qquad (99)$$

By using this tension control law it was evident that the amplitude of the out-of-plane swing angle could be significantly reduced during retrieval.

Boschitsch and Bendiksen [18] used a nonlinear control law based on optimal control theory to find retrieval histories that reduce the in-plane and out-of-plane librational motion of a tethered satellite. A different approach was used that specifies a given set of initial conditions and desired final condition together with a cost function that penalizes in-plane and out-of-plane deviations of the motion and uses the commanded tether length as the means of control.

It is sought to minimize a cost function i.e.

$$J = \phi[x(t_f), t_f] + \int_{t_o}^{t_f} \{ L[x(t), \mu(t), t]\} dt \tag{100}$$

by appropriate choice of the scalar control, $u(t)$, where t_o and t_f are given. $\phi\ [x(t_f), t_f]$ is the terminal cost and $L[x(t), u(t), t]$ is the Lagrangian. These are most common of the quadratic forms:

$$\phi\,[x(t_f),\ t_f] = \Sigma\ W_i (x_f - x_d)_i^2 \tag{101}$$

$$L[x(t),\ u(t),\ t] = \Sigma\, Q_i x_i^2(t) + R\ u^2(t) \tag{102}$$

where W_i, Q_i and R are weighting factors that indicate the importance of minimizing the associated state component, x_i, or control, u, and x_d is a desired final state such as the final desired tether length. With the implementation of a first order conjugate gradient method the optimal $\dot{L}$ control was obtained. The numerical simulation results showed that by suitable choice of state and control weightings the librational motion could be significantly reduced, and in-plane oscillations were more readily attenuated than those for the out-of-plane motion.

The more realistic model was not considered in this investigation such as including the transverse motion of the tether and the effect of atmospheric forces and eccentricity of the orbit.

In order to further improve the performance of the system an additional nonlinear tension control law was introduced by Liu Liangdong and Bainum [8] which is of the form:

$$T = -(m_s+m_t)\omega^2[F_c+FK_1\Delta L-FK_2\Delta L' + L(FK_3\alpha^2 + FK_4\gamma^2 + FK_5\gamma'^2)] \qquad (103)$$

where: m_s, m_t are the mass of subsatellite and tether, respectively; ω is the orbital frequency; ΔL and $\Delta L'$ are the difference between the tether length and some reference length (and its rate); the angles $\alpha, \gamma, \alpha, \gamma'$ are tether in-plane and out-of-plane swing angles and angular rates, respectively; and FK_i are optimal control law gains.

Because it was difficult to use strictly analytical methods to derive control gains for such nonlinear equations, the cost function was selected as below:

$$J_1 = [\int_0^{T_f} (\alpha^2+\gamma^2)dt]/T_f \qquad (104)$$

$$J_2 = [\int_0^{T_f} (B_1/L)^2 + (C_1/L)^2]dt/T_f \qquad (105)$$

$$J = J_1 + Q_2 J_2 \qquad (106)$$

where T_f, B_1, C_1 and Q_2 are the desired retrieval time, the first flexible modal (amplitude) state variables, and a weighting coefficient, respectively. According to the simulation results it was seen that the amplitude of the swing angles, α, γ (especially γ), were reduced greatly as compared with Rupp's [10] control law, and the transient responses were also better than those based on the control law in Ref. [17].

The tension control is unreliable during the terminal stage of retrieval when the tension becomes very small because of the small length of the tether. The tension might even become zero (slack tether) due to the longitudinal oscillations. To overcome this difficulty Banerjee and Kane [19] proposed that natural tether tension could be augmented with satellite-based, tether-aligned thrusters and that these thrusters would be capable of stabilizing and speeding up the retrieval process. The thruster augmented torque control was of the form.

$$T_c - T_{c_o} = K_\alpha \alpha + K_\alpha \alpha' + K_\gamma \gamma + K_\gamma \gamma' \qquad (107)$$

where T_c is the torque proportional to α' and γ'.

A summary of the control laws used by the various investigations is given in Table 3.

3.4 Recommendation Remarks

(1) Because most of the useful missions are carried out during the station keeping phase for Shuttle/Platform Tethered Subsatellite systems, the investigation of control laws should be focused mainly on the deployment and station keeping stages. Retrieval is less important than the first two for tether reflector applications where it may not be required to retrieve the tether (except possibly before rapid maneuvering). Although some nonlinear control laws were proposed (especially those which include thruster

augmentation of the tether tension control), it may still be difficult to implement an efficient and successful retrieval for the tethered reflector system.

(2) For deployment, the active tension modulation schemes proposed by Rupp[10] and subsequent investigators are more efficient than the purely passive scheme advocated by Kane and Levinson.

(3) Out of the various active control laws for proposed orbiting tethered reflector systems, Rupp's [10] control law is the most basic and is effective in controlling in-plane motion, but not adequate for out-of-plane motion control.

(4) Kissel's (Baker et al.)[12] tension control law, based on a combination of exponential and uniform rate of tether length, as a commanded length rate can be used both for in-plane and out-of-plane motion control during deployment. It was seen that the duration of time for in-plane deployment was reduced as compared with that in Rupp [10], and damping characteristics for both the case of station keeping and deployment were better than those for Rupp.

(5) Bainum and Kumar's [15] control law based upon the linear regulator problem of optimal control theory is suitable for adaptation with proposed Shuttle -Tethered systems, more specifically:

For station keeping purposes at altitudes where atmospheric effects are negligible, control laws result in improved transient response to initial perturbations as compared with previously developed control laws [Rupp's and

Kissel's]. The tether tension and power levels required for such control do not exceed previous requirements (for the TSS system).

For steady state station keeping requirements where atmospheric effects and eccentricity may be important, the optimal control law with gains based on optimal control theory can be used to bring the system to an in-plane equlibrium position and tension level which reflects a balance between the gravity-gradient and aerodynamics torques (forces).

For moderate duration deployment the same form of the optimal law, where the actual gains are adapted continuously to the commanded length, can result in improved damping (settling) characteristics with small amplitude initial excursions in the in-plane swing angle.

(6) The same form of the optimal control law as that in [15] can be used extensively in both 2-dimensional dynamic models [Bainum and Woodard] [4] and 3-dimensional dynamic models [Fan Ruying and Bainum] [5], of the Platform-Tethered-Sub-satellite systems during stationkeeping.

(7) When considering tether mass, tether flexibility, aero-dynamic force on the tether, and eccentricity of the orbit, i.e. a more complex dynamic model for tethered systems, Liu Liangdong and Bainum's alternate optimal control law made further improvements in the transient response of both the in-plane swing angle and vibration as compared with previous developed control laws such as in [Modi et al.[17]].

To sum up, control laws based upon optimal control theory offer the greatest potential for applications involving proposed orbiting tethered reflector systems.

Table 3 Summary of Tether Control Laws

Investigators	Motion Controlled	Type Control	Form of control law
Rupp [1975]	2d. rotations and stretch	tension control	$T = K_1 L + C_1 \dot{L} + K_2 L_c$
Kissel [Baker et al., 1976]	3d. rotations and stretch	tension control	$\frac{T}{m_s + \frac{1}{2} m_t} = (A^2+3)\,\omega^2 L + 2\xi A \omega \dot{L} - A^2 \omega^2 L_c$
Kalaghan [1978]	3d. rotations and stretch	tension control	Kissel law with lumped parameter model
Misra and Modi [1980]	2d. rotations, stretch and transverse vibration	length rate control	$L' = L_c' \,[1 + K_1^T X_1]$
Bainum and Kumar [1978]	3d. rotations and stretch	optimized tension control	$T - T_0 = K_\varepsilon \varepsilon + K_{\varepsilon'} \varepsilon' + K_\alpha \alpha + K_{\alpha'} \alpha'$

Investigators	Motion controlled	Type Control	Form of Control Law
* Bainum Woodard Juang [1985]	2d. rotations and stretch	optimized tension, momentum control	$T_L = K_{\varepsilon\varepsilon}\varepsilon + K_{\varepsilon\varepsilon'}\varepsilon' + K_{\varepsilon\alpha_v}\alpha_v + K_{\varepsilon\alpha_v'}\alpha_v' + K_{\varepsilon\theta}\theta_v + K_{\varepsilon\theta_v'}\theta_v'$ $T_\theta = K_{\theta\varepsilon}\varepsilon + K_{\theta\varepsilon'}\varepsilon' + K_{\theta\alpha_v}\alpha_v + K_{\theta\alpha_v'}\alpha_v' + K_{\theta\theta_v}\theta_v + K_{\theta\theta_v'}\theta_v'$
* Fan Ruying Bainum [1988]	3d. rotations and stretch	optimized tension, momentum control	$T_L = K_{\varepsilon\varepsilon}\varepsilon + K_{\varepsilon\theta_v}\theta_v + K_{\varepsilon\alpha_v}\alpha_v + K_{\varepsilon\varepsilon'}\varepsilon' + K_{\varepsilon\theta_v'}\theta_v' + K_{\varepsilon\alpha_v'}\alpha_v'$ $T_\theta = K_{\theta\varepsilon}\varepsilon + K_{\theta\theta_v}\theta_v + K_{\theta\alpha_v}\alpha_v + K_{\theta\varepsilon'}\varepsilon' + K_{\theta\theta_v'}\theta_v' + K_{\theta\alpha_v'}\alpha_v'$ $T_\phi = K_{\phi\phi}\phi + K_{\phi\psi}\psi + K_{\phi\Upsilon}\Upsilon + K_{\phi\phi'}\phi' + K_{\phi\psi'}\psi' + K_{\phi\Upsilon'}\Upsilon'$ $T_\psi = K_{\psi\phi}\phi + K_{\psi\psi}\psi + K_{\psi\Upsilon}\Upsilon + K_{\psi\phi'}\phi' + K_{\psi\psi'}\psi' + K_{\psi\Upsilon'}\Upsilon'$
Xu et al. [1981]	3d. rotations	nonlinear length rate control	$\frac{L'}{L} = K_\theta[1 + K_{\alpha'}\alpha' + K_{\Upsilon'}\Upsilon'^2]$
Modi et al. [1981]	3d. rotations and stretch	nonlinear tension control	$T - T_0 = K_L L + K_{L'} L' + K_{\Upsilon'}\Upsilon'^2$
Bosehitsch Bendiksen [1988]	3d. rotations and stretch	optimal $\dot{\ell}$ control	$X = (\alpha\ \alpha'\ \Upsilon\ \Upsilon'\ L)^T$ $J = \phi[X(t_f), t_f] + \int_{t_0}^{t_f} \Sigma Q_i x_i^2(t) + R \cdot U^2(t)\, dt$

Investigators	Motion controlled	Type Control	Form of Control Law
Liu Liangdong Bainum [1987]	3d. rotations, stretch and transverse vibrations	optimal tension control	$T^{*} = -(K_{\varepsilon}\varepsilon + K_{\varepsilon'}\varepsilon' + K_{\alpha}\alpha + K_{\alpha'}\alpha' + K_{\eta_1}\eta_1 + K_{\eta_1'}\eta_1')$ $T^{**} = -(m_s + m_t)\,\omega^2 [F_c + FK_1\Delta L + FK_2\Delta L' + L(FK_3\alpha^2 + FK_4\gamma^2 + FK_5\gamma'^2)]$ T^{*}: for station keeping ($F_c = L_c'' + 3\frac{(m_s + m_t/2)}{m_s + m_t}L_c$; T^{**}: for retrieval $L_c = L_o e^{-pt}$)
Banerjee Kane [1982]	3d. rotations and stretch	thrust augmented torque control	$T_c - T_{co} = K_{\alpha}\alpha + K_{\alpha'}\alpha' + K_{\gamma}\gamma + K_{\gamma'}\gamma'$
* Lakshmanan Modi Misra [1987]	3d. rotations and stretch	optimized tension, thruster and offset control	$u = \begin{bmatrix} M_{\alpha} \\ M_{\beta} \\ M_{\gamma} \\ T_e \end{bmatrix}$ $u = \begin{bmatrix} M_{\alpha} \\ M_{\beta} \\ M_{\gamma} \\ T_{\alpha} \\ T_{\gamma} \\ T_{\eta} \end{bmatrix}$ $u = \begin{bmatrix} M_{\alpha} \\ M_{\beta} \\ M_{\gamma} \\ D_{x,p} \\ D_{y,p} \\ D_{z,p} \end{bmatrix}$

4. NONLINEAR DYNAMIC EQUATIONS OF A TETHERED ANTENNA/REFLECTOR IN ORBIT

4.1 Introduction

The linearized equations of the orbiting tethered antenna system have been obtained in Chap. II, and active tether control laws (LQR) based on the linear model have been efficiently used to apply active control during stationkeeping of the system. However, for the deployment abd retrieval, the linear system model may not represent the physical situation accurately any more and the active control laws based on this model certainly may not be as effective during deployment and retrieval as during stationkeeping. This is due to the large slewing angles and inherent instability of the out-of-plane motion of the tether. Furthermore, second order terms in length rate are directly coupled with out-of-plane modal amplitude terms. Hence to damp the out-of-plane motion using length rate control (tension control) and to simulate the dynamic behavior of the system during the deployment and retrieval, it is necessary to use the nonlinear equations.

The general dynamics of a tethered system is rather complex and hence, early dynamical models were based on a number of simplfying assumptions. An overview of the development in this area, particuraly system models and proposed control laws, has been given by Misra and Modi [14] and Bainum and Kumar [15]. The system models have grown from initial massless idealized tether models to complex representations encompassing all the tether vibrations (flexibility) and end body motions, as exemplied in the model by Misra and Modi [14].

As for the tethered antenna/reflector system, the translational motions of the subsatellite and transverse vibrations of the tether will affect the rigid body motion of the orbiting antenna; therefore, all these effects will be included in the formulation of the nonlinear system equations for the further study of simulating the system dynamic be havior and applying active control during the deployment and retrieval.

4.2 The Assumptions

The following assumptions are made to develop the model equations:

a). The antenna mass is far greater than that of the tether and the subsatellite; consequently, the center of mass of the system may be taken to coincide with that of the antenna.

b). The tether is assumed uniform with a constant mass per length.

c). Longitudinal stretching is not considered , and longitudinal vibration is neglected compared with the transverse vibrations.

d). The shell is considered to be a rigid body.

e). The subsatellite is considered to be a point mass.

f). No random inputs or unknown disturbances are considered.

g). Only first order gravity-gradient effects are considered and the orbit is assumed circular.

4.3 Kinematics of the System

The coordinate systems used in the development of the system equations of motion are shown in Fig. 7. $O_p X_o Y_o Z_o$ is an orbit-reference centered at the center of mass of the shell, O_p , with $O_p X_o$ along the local vertical and $O_p Y_o$ along the orbital angular velocity direction, $O_p Z_o$ along the orbital tangent velocity direction.

$O_p X_p Y_p Z_p$ is a shell body reference frame, R_p , where $O_p X_p$, $O_p Y_p$, $O_p Z_p$ are principal axes of the shell. $OX_t Y_t Z_t$ is the subsatellite undeformed tether reference frame, R_t , with OX_t along the undeformed tether line, where O is the point from which the tether is deploying or retrieving. The coordinates of O in the shell frame are (h, 0, 0).

The Euler angles ψ, θ, ϕ are the yaw, pitch and roll angles of the shell, respectively. α, γ are pitch (in-plane) and yaw (out-of-plane) angles of the tether.

For convenience, the transverse vibrations of the tether are expanded in terms of a set of admissible functions.

$$v=\Sigma\Phi_n(x)B_n(t) \qquad w=\Sigma\phi_n(x)B_n(t) \tag{108}$$

where $\phi_n(x)=\sin(n\pi x/L)$, v--out-of-plane displacement of the tether, w--in-plane displacement.

Therefore, the whole system has the following degrees of freedom:

ψ, θ, ϕ ---rigid body motion of the shell.

α, γ ---translational motion of the subsatellite.

L ---length of the tether.

B_n, C_n ---transverse vibrations of the flexible tether.

The transformation matrices from $O_p X_p Y_p Z_p$ to $O_o X_o Y_o Z_o$ and $O X_t Y_t Z_t$ to $O_p X_p Y_p Z_p$ are given by

$$M_p = \begin{bmatrix} C\theta\, C\phi & -C\theta\, S\phi & S\theta \\ -S\psi S\theta C\phi + C\psi S\phi & S\psi S\theta S\phi + C\psi C\phi & S\psi C\theta \\ C\psi S\theta\, C\phi + S\psi S\phi & -C\psi S\theta\, S\phi + S\psi C\phi & C\psi C\theta \end{bmatrix} \tag{109}$$

$$\begin{bmatrix} x_o \\ y_o \\ z \end{bmatrix} = M_p \begin{bmatrix} x_p \\ y_p \\ z_p \end{bmatrix} \tag{110}$$

$$M_t = \begin{bmatrix} C\alpha C\gamma & -C\alpha S\gamma & -S\alpha \\ S\gamma & C\gamma & 0 \\ S\alpha C\gamma & -S\alpha S\gamma & C\alpha \end{bmatrix} \tag{111}$$

$$\begin{bmatrix} x_p \\ y_p \\ z_p \end{bmatrix} = \begin{bmatrix} h \\ o \\ o \end{bmatrix} + M_t \begin{bmatrix} x_t \\ y_t \\ z_t \end{bmatrix} \qquad (112)$$

where C -- cos, S -- sin

The angular velocity of the shell is given by

$$\overline{\omega}_p = \Omega_x \hat{i}_p + \Omega_y \hat{j}_p + \Omega_z \hat{k}_p \qquad (113)$$

where

$$\Omega_x = \dot{\psi}\, C\theta\, C\phi + \dot{\theta}\, S\phi + \omega_c\, (- S\psi\, S\theta\, C\phi + C\psi\, S\phi)$$

$$\Omega_y = \dot{\theta}\, C\phi - \dot{\psi}\, C\theta\, S\phi + \omega_c\, (S\psi\, S\theta\, S\phi + C\psi\, C\phi) \qquad (114)$$

$$\Omega_z = \dot{\phi} + \dot{\psi}\, S\theta - \omega_c\, S\psi\, C\theta$$

The angular velocity of the tether is given by

$$\overline{\omega}_t = \omega_x \hat{i}_t + \omega_y \hat{j}_t + \omega_z \hat{k}_t \qquad (115)$$

$$\omega_x = C\alpha\, C\gamma\, \Omega_x + S\gamma\, \Omega_y + S\alpha\, C\gamma\, \Omega_z + S\gamma\, \dot{\alpha}$$

$$\omega_y = - C\alpha\, S\gamma\, \Omega_x + C\gamma\, \Omega_y - S\alpha\, S\gamma\, \Omega_z + C\gamma\, \dot{\alpha} \qquad (116)$$

$$\omega_z = - S\alpha\, \Omega_x + C\alpha\, \Omega_z + \dot{\gamma}$$

4.4 Dynamic Equations of the System

4.4.1 Rigid Body Motion of the Shell

The Euler–Newtonian method is used to develop the dynamical equations of the shell motion.

The angular momentum of the shell is given by

$$\overline{N} = I_{xx}\Omega_x \hat{i}_p + I_{yy}\Omega_y \hat{j}_p + I_{zz}\Omega_z \hat{k}_p \qquad (117)$$

The time derivative of the N is written as

$$\dot{\overline{N}} = [\, I_{xx}\Omega_x - (I_{yy} - I_{zz})\Omega_y \Omega_z]\hat{i}_p + [\, I_{yy}\Omega_y - (I_{zz} - I_{xx})\Omega_x \Omega_z]\hat{j}_p + [\, I_{zz}\Omega_z - (I_{xx} - I_{yy})\Omega_x \Omega_y]\hat{k}_p \qquad (118)$$

There are two forces acting on the shell, one is the gravitational force, the other tether tension force, $\overline{T}$. Therefore, the torque exerted on the shell is

$$\bar{L} = \bar{L}_G + \bar{L}_T \tag{119}$$

where $\bar{L}_G$ is the gravitational torque and $\bar{L}_T$ is the torque of the tether tension force.

$$\bar{L}_T = -h'T\sin\alpha\,\cos\gamma\,\hat{j}_P + h'T\sin\gamma\,\hat{k}_P \tag{120}$$

Here, we only consider first-order gravitational forces. It is well known that the first-order gravitational force acting on any point $P(x,y,z)$ in the shell can be expressed [7] in the shell frame as:

$$d\bar{f} = \omega_c^2 dm \begin{bmatrix} 3\,C\theta^2 C\phi^2 - 1 & -3\,C\theta^2 S\phi\,C\phi & -3\,S\theta\,C\theta\,C\phi \\ -3\,C\theta^2 S\phi\,C\phi & 3\,C\theta^2 S\phi^2 - 1 & 3\,S\theta\,C\theta\,S\phi \\ -3\,S\theta\,C\theta\,C\phi & 3\,S\theta\,C\theta\,S\phi & 3\,S\theta^2 - 1 \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix} \tag{121}$$

Hence, the torque of the gravitational force

$$\bar{L}_G = \int \bar{r} \times d\bar{f} = \omega_c^2 \begin{bmatrix} 3\,S\theta\,C\theta\,S\phi\,(I_{yy} - I_{zz}) \\ -3\,S\theta\,C\theta\,C\phi\,(I_{zz} - I_{xx}) \\ -3\,C\theta^2 S\phi\,C\phi\,(I_{xx} - I_{yy}) \end{bmatrix} \tag{122}$$

According to the Euler–Newtonian equation, we obtain

$$\begin{aligned} I_{xx}\dot{\Omega}_x - (I_{yy} - I_{zz})\,\Omega_y\,\Omega_z &= 3\,\omega_c^2\,S\theta\,C\theta\,S\phi\,(I_{yy} - I_{zz}) \\ I_{yy}\dot{\Omega}_y - (I_{zz} - I_{xx})\,\Omega_x\,\Omega_z &= -h'TS\alpha\,C\gamma - 3\,\omega_c^2\,S\theta\,C\theta\,C\phi\,(I_{zz} - I_{xx}) \\ I_{zz}\dot{\Omega}_z - (I_{xx} - I_{yy})\,\Omega_x\,\Omega_y &= h'TS\gamma - 3\,\omega_c^2\,C\theta^2 S\phi\,C\phi\,(I_{xx} - I_{yy}) \end{aligned} \tag{123}$$

The above are the nonlinear dynamical equations for the shell.

4.4.2 Translational Motion of the Subsystem (tether and subsatellite)

The Langrange approach is used to develop the dynamical equations of the translational motions of the subsytem. Therefore, we need to calculate the kinetic and potential energy of the tether and the subsatellite.

A). KInetic Energy

The velocity of any point Q(x,y,z) on the tether is of the following form:

$$\bar{V} = \bar{V}_o + \bar{\omega}_t \times \bar{r}_t + \dot{\bar{r}}_t \tag{124}$$

where

$$\bar{V}_o = h \left\{ \begin{bmatrix} -S\alpha\, C\gamma\, \Omega_y + S\gamma\, \Omega_z \\ S\alpha\, S\gamma\, \Omega_y + C\gamma\, \Omega_z \\ -C\alpha\, \Omega_y \end{bmatrix} + \begin{bmatrix} \dot{L} \\ 0 \\ 0 \end{bmatrix} \right\} \begin{bmatrix} \hat{i}_t \\ \hat{j}_t \\ \hat{k}_t \end{bmatrix} \tag{125}$$

$$\bar{\omega}_t \times \bar{r}_t = \left\{ \begin{bmatrix} 0 & w & -v \\ -w & 0 & x \\ v & -x & 0 \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \right\} \begin{bmatrix} \hat{i}_t \\ \hat{j}_t \\ \hat{k}_t \end{bmatrix} \tag{126}$$

$$\dot{\bar{r}}_t = \begin{bmatrix} 0 \\ \dot{v} \\ \dot{w} \end{bmatrix}^T \begin{bmatrix} \hat{i}_t \\ \hat{j}_t \\ \hat{k}_t \end{bmatrix} \tag{127}$$

Hence, the kinetic energy of the tether is given by

$$T_t = \frac{1}{2}\int_0^L \rho\, |\bar{V}|^2 dx = \frac{1}{2}\int_0^L \rho\, |\bar{V}_o|^2 dx + \frac{1}{2}\int_0^L \rho\, |\bar{\omega}_t \times \bar{r}_t|^2 dx + \frac{1}{2}\int_0^L \rho\, |\dot{\bar{r}}_t|^2 dx$$
$$+ \int_0^L \rho\, \bar{V}_o \cdot (\bar{\omega}_t \times \bar{r}_t) dx + \int_0^L \rho\, \bar{V}_o \cdot \dot{\bar{r}}_t\, dx + \int_0^L \rho\, (\bar{\omega}_t \times \bar{r}_t) \cdot \dot{\bar{r}}_t\, dx \tag{128}$$

where

$$T_1 = \frac{1}{2}\int \rho\, |\bar{V}_o|^2 dx = \frac{m_t}{2} \left[h^2 \Omega_y^2 + h^2 \Omega_z^2 + \dot{L}^2 + 2h\dot{L} \left(\sin\gamma\, \Omega_z - \sin\alpha\, \cos\gamma\, \Omega_y \right) \right] \tag{129}$$

$$T_2 = -\frac{1}{2}\int\rho\,|\bar{\omega}_t \times \bar{r}_t|^2 dx = -\frac{1}{2}\,\omega_t^T \int\rho \begin{bmatrix} v^2+w^2 & -xv & -xw \\ -xv & x^2+w^2 & -vw \\ -xw & -vw & x^2+v^2 \end{bmatrix} dx \;\; \omega_t$$

$$= -\frac{m_t}{2}\Big[\frac{1}{2}\Sigma(B_n^2 + C_n^2)\omega_x^2 + \Big(\frac{L^2}{3} + \frac{1}{2}\Sigma C_n^2\Big)\omega_y^2 + \Big(\frac{L^2}{3} + \frac{1}{2}\Sigma B_n^2\Big)\omega_z^2$$

$$+ \frac{2L}{\pi}\Sigma\frac{(-1)^n}{n} B_n\omega_x\omega_y + \frac{2L}{\pi}\Sigma\frac{(-1)^n}{n} C_n\omega_x\omega_z - \Sigma B_n C_n\omega_y\omega_z\Big] \qquad (130)$$

$$T_3 = \frac{1}{2}\int\rho(|\dot{v}|^2 + |\dot{w}|^2)dx$$

where

$$\dot{v} = \Sigma \sin\frac{n\pi x}{L}\dot{B}_n - \frac{\pi\dot{L}}{L^2}\Sigma\, n\,x\cos\frac{n\pi x}{L}\,B_n \qquad (131)$$

$$\dot{w} = \Sigma \sin\frac{n\pi x}{L}\dot{C}_n - \frac{\pi\dot{L}}{L^2}\Sigma\, n\,x\cos\frac{n\pi x}{L}\,C_n$$

$$T_4 = \int\rho\,\bar{V}_0\cdot(\bar{\omega}_t \times \bar{r}_t)\,dx = V_0^T\int\rho\begin{bmatrix} 0 & w & -v \\ -w & 0 & x \\ v & -x & 0 \end{bmatrix} dx \;\; \omega_t$$

$$= m_t\Big\{-\frac{h}{\pi}\,\Omega_y\omega_x\,\Big[\sin\alpha\sin\gamma\;\Sigma\frac{1+(-1)^{n+1}}{n}C_n + \cos\alpha\;\Sigma\frac{1+(-1)^{n+1}}{n}B_n\Big]$$

$$-\frac{h}{\pi}\Omega_z\,\omega_x\cos\gamma\,\Sigma\frac{1+(-1)^{n+1}}{n}C_n + h\,\Omega_y\omega_y\Big[-\frac{L}{2}\cos\alpha - \frac{L}{\pi}\sin\alpha\cos\gamma\;\Sigma\frac{1+(-1)^{n+1}}{n}C_n\Big]$$

$$+\frac{h}{\pi}\,\Omega_z\omega_y\sin\gamma\;\Sigma\frac{1+(-1)^{n+1}}{n}C_n + h\,\Omega_y\omega_z\,\Big[-\frac{L}{2}\sin\alpha\sin\gamma + \frac{L}{\pi}\sin\alpha\cos\gamma\,\Sigma\frac{1+(-1)^{n+1}}{n}B_n\Big]$$

$$+ h\,\Omega_z\omega_z\,\Big[-\frac{L}{2}\cos\gamma - \frac{L}{\pi}\sin\gamma\,\Sigma\frac{1+(-1)^{n+1}}{n}B_n\Big] + \frac{L}{\pi}\Big[\Sigma\frac{1+(-1)^{n+1}}{n}C_n\,\omega_y$$

$$-\Sigma\frac{1+(-1)^{n+1}}{n}B_n\,\omega_z\Big]\Big\} \qquad (132)$$

$$T_5 = \int \rho \bar{V}_o \cdot \dot{\bar{r}}_t \, dx = \frac{m_t h}{\pi} \Omega_y [\, \sin\alpha \sin\gamma \, \Sigma \frac{1+(-1)^{n+1}}{n} \dot{B}_n - \cos\alpha \, \Sigma \frac{1+(-1)^{n+1}}{n} \dot{C}_n +$$

$$\sin\alpha \sin\gamma \left(-\frac{\dot{L}}{L} \right) \Sigma \frac{1+(-1)^{n+1}}{n} B_n - \cos\alpha \left(-\frac{\dot{L}}{L} \right) \Sigma \frac{1+(-1)^{n+1}}{n} C_n \,] \qquad (133)$$

$$+ \frac{m_t h}{\pi} \cos\gamma \, \Omega_z \, [\, \Sigma \frac{1+(-1)^{n+1}}{n} \dot{B}_n + \left(-\frac{\dot{L}}{L} \right) \Sigma \frac{1+(-1)^{n+1}}{n} B_n \,]$$

$$T_6 = \int \rho \, (\bar{\omega}_t \times \bar{r}_t) \cdot \dot{\bar{r}}_t \, dx = \omega_t^T \int \rho \begin{bmatrix} 0 & -w & v \\ w & 0 & x \\ -v & x & 0 \end{bmatrix} \begin{bmatrix} 0 \\ \dot{v} \\ \dot{w} \end{bmatrix} dx$$

$$= \frac{m_t}{2} \Sigma (B_n \dot{C}_n - C_n \dot{B}_n) \omega_x + \frac{m_t L}{\pi} [\, \Sigma \frac{(-1)^n}{n} \dot{C}_n + 2 \left(\frac{\dot{L}}{L} \right) \Sigma \frac{(-1)^n}{n} C_n \,] \, \omega_y$$

$$- \frac{m_t L}{\pi} [\, \Sigma \frac{(-1)^n}{n} \dot{B}_n + 2 \left(\frac{\dot{L}}{L} \right) \Sigma \frac{(-1)^n}{n} B_n \,] \, \omega_z \qquad (134)$$

The velocity of the subsatellite

$$\bar{V}_s = \left\{ h \, \Omega_y \begin{bmatrix} -S\alpha \, C\gamma \\ S\alpha \, S\gamma \\ -C\alpha \end{bmatrix} + h \, \Omega_z \begin{bmatrix} S\gamma \\ C\gamma \\ 0 \end{bmatrix} + \begin{bmatrix} \dot{L} \\ 0 \\ 0 \end{bmatrix} + L \begin{bmatrix} 0 \\ \omega_z \\ -\omega_y \end{bmatrix} \right\} \begin{bmatrix} \hat{i}_t \\ \hat{j}_t \\ \hat{k}_t \end{bmatrix} \qquad (135)$$

Hence, the kinetic energy of the subsatellite

$$T_s = \frac{1}{2} m_s |\bar{V}_s|^2 = \frac{1}{2} m_s [\, h^2 \Omega_y^2 + h^2 \Omega_z^2 + \dot{L}^2 + L^2 \omega_z^2 + L^2 \omega_y^2 - 2h \sin\alpha \cos\gamma \, \Omega_y \dot{L}$$

$$+ 2h \sin\gamma \, \Omega_z \dot{L} + 2hL \Omega_y (\cos\alpha \, \Omega_y + \sin\alpha \sin\gamma \, \omega_z) + 2hL \cos\gamma \, \Omega_z \omega_z \,] \qquad (136)$$

B). Elastic Potential Energy of the Tether

Suppose the tether is an Euler–Berhou'i beam. Therefore, the elastic potential energy is of the following form:

$$E = \frac{1}{2}\int EI\,(v_{xx}^2 + w_{xx}^2)\,dx = \frac{\pi^4 EI}{4L^3}\Sigma(B_n^2 + C_n^2) \qquad (137)$$

C). First-order Gravity Gradient Field

Since the origin of the orbital reference frame moves along a free-fall trajectory, the only gravitational forces acting on the subsystem arise from the gravity-gradient field. The gravity-gradient force terms are obtained by a Taylor-series expansion of the gravity field about the free-fall trajectory. The first-order terms of this series are well known. Since $B_n / L \ll 1$, $C_n / L \ll 1$, the effect of the transverse vibration of the tether on the gravity force is neglected. Applying these terms to a mass particle of size dm results in the following:

$$\begin{aligned} dF_x &= 2\,\omega_c^2\, x\; dm \\ dF_y &= -\,\omega_c^2\, y\; dm \qquad (138) \\ dF_z &= -\,\omega_c^2\, z\; dm \end{aligned}$$

where (x, y, z) are the coordinates of the particle dm in the orbital reference frame, $\bar{\omega}_c$ is the orbital angular velocity.

Summing up the forces over all mass particles of the dynamical system yields the first-order gravity-gradient terms as:

$$\begin{aligned} Q_\alpha = {} & 3\left(\frac{m_t}{3} + m_s\right)\omega_c^2 L^2 [\, S\alpha\, C\alpha\, C^2\gamma\, (S^2\theta - C^2\theta\, C^2\phi) - C^2\alpha\, C^2\gamma\, S\theta\, C\theta\, C\phi\, + \\ & S\alpha\, S\gamma\, C\gamma\, C^2\theta\, S\phi\, C\phi + C\alpha\, S\gamma\, C\gamma\, S\theta\, C\theta\, S\phi \,] - \left(\frac{m_t}{2} + m_s\right)\omega_c^2 h L\, [\\ & S\alpha\, C\gamma\, (3\, C^2\theta\, C^2\phi - 1) + 3\, C\alpha\, C\gamma\, S\theta\, C\theta\, C\phi \,] \qquad (139) \end{aligned}$$

$$Q_\gamma = 3(\frac{m_t}{3} + m_s)\omega_c^2 L^2 \, [\, S\gamma \, C\gamma \, (C\alpha^2 \, C\theta^2 S\phi^2 - C\theta^2 C\phi^2) + S\alpha \, C\alpha \, S2\gamma \, S\theta \, C\theta \, C\phi \, +$$

$$S\alpha \, C2\gamma \, S\theta \, C\theta \, S\phi - C\alpha \, C2\gamma \, C\theta^2 S\phi \, C\phi \,] - (\frac{m_t}{2} + m_s)\omega_c^2 h L [\, C\alpha \, S\gamma \, (3 \, C\theta^2 \, C\phi^2 - 1)$$

$$+ 3 \, C\gamma \, C\theta^2 S\phi \, C\phi - 3 \, S\alpha \, S\gamma \, S\theta \, C\theta \, C\phi \,] \qquad (140)$$

$$Q_L = (\frac{m_t}{2} + m_s) \, \omega_c^2 L [\, C\alpha^2 \, C\gamma^2 (3C\theta^2 \, C\phi^2 - 1) + S\gamma^2 (3C\theta^2 \, S\phi^2 - 1) + S\alpha^2 \, C\gamma^2 (3S\theta^2 - 1)$$

$$- 6 \, S\alpha \, C\alpha \, C\gamma^2 S\theta \, C\theta \, C\phi + 6 \, S\alpha \, S\gamma \, C\gamma \, S\theta \, C\theta \, S\phi - 6 \, C\alpha \, S\gamma \, C\gamma \, C\theta^2 \, S\phi \, C\phi \qquad (141)$$

$$+ (m_t + m_s)\omega_c^2 h \, [\, C\alpha \, C\gamma \, (3C\theta^2 \, C\phi^2 - 1) - 3 \, S\gamma \, C\theta^2 S\phi \, C\phi - 3 \, S\alpha \, C\gamma \, S\theta \, C\theta \, C\phi \,)]$$

Since the antenna mass is far greater than that of the tether and subsatellite, it is assumed that the Euler angles of rigid body motions of the satellite and their rates, the vibrating mode shape and their rates are small (first order terms). that is , ψ, θ, ϕ, Ω_x , Ω_y, Ω_z B_n, C_n, $\dot{B}_n$, $\dot{C}_n$ $\ll$ 1 . Omitting the third order terms and above, we obtain the following dynamical equations:

In plane motion

$$
\begin{aligned}
&\Big[-\Big(\frac{m_t}{3}+m_s\Big)L^2\cos\alpha\sin\gamma\cos\gamma+\frac{m_tL}{\pi}\cos\alpha\cos2\gamma\,\Sigma\,\frac{(-1)^n}{n}B_n\Big]\dot{\Omega}_x \\
&+\Big[\Big(\frac{m_t}{3}+m_s\Big)L^2\cos\gamma^2+\Big(\frac{m_t}{2}+m_s\Big)hL\cos\alpha\cos\gamma\Big]\dot{\Omega}_y \\
&+\Big[-\Big(\frac{m_t}{3}+m_s\Big)L^2\sin\alpha\sin\gamma\cos\gamma+\frac{m_tL}{\pi}\cos\alpha\sin\gamma\,\Sigma\,\frac{(-1)^n}{n}C_n\Big]\dot{\Omega}_z \\
&+\Big(\frac{m_t}{3}+m_s\Big)L^2\cos\gamma^2\,\ddot{\alpha}+\frac{m_tL}{\pi}\sin\alpha\,\Sigma\,\frac{(-1)^n}{n}C_n\,\ddot{\gamma}+\frac{m_t\cos\gamma}{\pi}\,\Sigma\,\frac{1+(-1)^n}{n}C_n\,\ddot{L} \\
&+\frac{m_tL\cos\gamma}{\pi}\,\Sigma\,\frac{(-1)^n}{n}\ddot{C}_n-\Big(\frac{m_t}{3}+m_s\Big)L^2\sin\alpha\cos\alpha\cos\gamma^2\,\Omega_x^2 \\
&+\Big[-\Big(\frac{m_t}{3}+m_s\Big)L^2\sin\alpha\sin\gamma\cos\gamma+\frac{m_tL}{\pi}\Big(\cos\alpha\sin\gamma\,\Sigma\,\frac{(-1)^n}{n}C_n \\
&+\sin\alpha\cos\gamma^2\,\Sigma\,\frac{(-1)^n}{n}B_n\Big)\Omega_x\Omega_y+\Big[\Big(\frac{m_t}{3}+m_s\Big)L^2\cos2\alpha\cos\gamma^2 \\
&+\Big(\frac{m_t}{2}+m_s\Big)hL\cos\alpha\cos\gamma\Big]\Omega_x\Omega_z+\Big[\Big(\frac{m_t}{2}+m_s\Big)hL\sin\alpha\cos\gamma^2 \\
&+\frac{m_th}{\pi}\cos\alpha\,\Sigma\,\frac{1+(-1)^{n+1}}{n}C_n\Big]\Omega_y^2+\Big[\Big(\frac{m_t}{3}+m_s\Big)L^2\cos\alpha\sin\gamma\cos\gamma \\
&-\frac{m_tL}{\pi}\cos\alpha\cos2\gamma\,\Sigma\,\frac{(-1)^n}{n}B_n\Big]\Omega_y\Omega_z+\Big[\Big(\frac{m_t}{3}+m_s\Big)L^2\sin\alpha\cos\alpha\cos\gamma^2 \\
&+\Big(\frac{m_t}{2}+m_s\Big)hL\sin\alpha\cos\gamma\Big]\Omega_z^2+2\Big(\frac{m_t}{3}+m_s\Big)L^2\cos\alpha\sin\gamma^2\,\Omega_x\dot{\gamma} \\
&-\frac{h}{\pi}\sin\alpha\sin\gamma\,\Sigma\,\frac{1+(-1)^{n+1}}{n}B_n\Omega_y\dot{\alpha}-2\Big[\Big(\frac{m_t}{3}+m_s\Big)L^2\sin\gamma\cos\gamma \\
&+\Big(\frac{m_t}{2}+m_s\Big)hL\cos\alpha\sin\gamma-\frac{m_tL}{\pi}\cos2\gamma\,\Sigma\,\frac{(-1)^n}{n}B_n \\
&+\frac{m_th}{\pi}\cos\alpha\cos\gamma\,\Sigma\,\frac{1+(-1)^{n+1}}{n}B_n\Big]\Omega_y\dot{\gamma}-2\Big[\Big(\frac{m_t}{3}+m_s\Big)L^2\sin\gamma\cos\gamma \\
&-\frac{m_tL}{\pi}\cos2\gamma\,\Sigma\,\frac{(-1)^n}{n}B_n\Big]\dot{\alpha}\dot{\gamma}+\frac{m_tL}{\pi}\cos\gamma\,\Sigma\,\frac{(-1)^n}{n}\dot{C}_n\dot{\gamma}+\frac{m_t\cos\gamma}{\pi}\,\Sigma\,\frac{1+(-1)^n}{n}\dot{C}_n\dot{L} \\
&-\Big[2\Big(\frac{m_t}{2}+m_s\Big)L\cos\alpha\sin\gamma\cos\gamma-\frac{2m_t}{\pi}\cos\alpha\cos2\gamma\,\Sigma\,\frac{(-1)^n}{n}B_n \\
&+\frac{m_t}{\pi}\cos\alpha\,\Sigma\,\frac{1+(-1)^{n+1}}{n}B_n\Big]\Omega_x\dot{L}+2\Big[\Big(\frac{m_t}{2}+m_s\Big)L\cos\gamma^2 \\
&+(m_t+m_s)h\cos\alpha\cos\gamma\Big]\Omega_y\dot{L}-2\Big(\frac{m_t}{2}+m_s\Big)L\sin\alpha\sin\gamma\cos\gamma\,\Omega_z\dot{L}
\end{aligned}
$$

$$+ 2\left(\frac{m_t}{2} + m_s\right)L\cos^2\gamma\, \dot{\alpha}\, \dot{L} - \left(\frac{m_t \sin\gamma}{\pi}\Sigma \frac{1+3(-1)^{n+1}}{n} C_n + \frac{\rho}{2}\cos\gamma\, \Sigma B_n C_n \right)\dot{\gamma}\, \dot{L}$$

$$+ \frac{m_t \cos\gamma}{\pi}\, \Sigma \frac{1+3(-1)^n}{n} \dot{C}_n \dot{L} - \frac{2\, m_t L}{\pi}\cos\alpha\, \sin^2\gamma\, \Sigma \frac{(-1)^n}{n} \dot{B}_n\, \Omega_x$$

$$+\left(\frac{m_t L}{\pi} \sin2\gamma\, \Sigma \frac{(-1)^n}{n} \dot{B}_n - \frac{2\, m_t h}{\pi}\cos\alpha\, \sin\gamma\, \Sigma \frac{1+(-1)^{n+1}}{n} \dot{B}_n \right)\Omega_y$$

$$- \frac{2\, m_t L}{\pi} \sin\alpha\, \sin^2\gamma\, \Sigma \frac{(-1)^n}{n} \dot{B}_n\, \Omega_z + \frac{2\, m_t L}{\pi}\sin\gamma \cos\gamma\, \Sigma \frac{(-1)^n}{n} \dot{B}_n\, \dot{\alpha}$$

$$+ \frac{2\, m_t L}{\pi}\cos\alpha\, \sin\gamma\, \Sigma \frac{(-1)^n}{n}\dot{C}_n\, \dot{\gamma} = Q_\alpha \qquad (142)$$

Out-of-plane motion

$$\left[-\left(\frac{m_t}{3} + m_s \right) L^2 \sin\alpha + \frac{m_t L}{\pi} \cos\alpha\, \cos\gamma\, \Sigma \frac{(-1)^n}{n} C_n \right] \dot{\Omega}_x + \left(\frac{m_t}{2} + m_s\right)hL\cos\gamma\, \dot{\Omega}_z$$

$$+\left(\frac{m_t}{3} + m_s\right)L^2\, \ddot{\gamma} + \left(\frac{m_t L}{\pi} \sin\gamma\, \Sigma \frac{(-1)^n}{n} C_n - \frac{m_t \cos\gamma}{2}\Sigma B_n C_n \right) \ddot{\alpha}$$

$$- \frac{m_t}{\pi}\Sigma \frac{1+(-1)^n}{n} B_n \ddot{L} - \frac{m_t L}{\pi}\Sigma \frac{(-1)^n}{n} \ddot{B}_n + \left[\left(\frac{m_t}{3} + m_s \right)L^2\cos\alpha\, \cos2\gamma\right.$$

$$\left. + \left(\frac{m_t}{2} + m_s\right)hL\, \cos^2\alpha\, \cos\gamma \right] \Omega_x \Omega_y + \left[\left(\frac{m_t}{3} + m_s\right)L^2 \sin\gamma\, \cos\gamma\right.$$

$$+ \left(\frac{m_t}{2} + m_s\right)hL\, \cos\alpha\, \sin\gamma - \frac{m_t L}{\pi} \cos2\gamma\, \Sigma \frac{(-1)^n}{n} B_n$$

$$\left. + \frac{m_t h}{\pi} \cos\alpha\, \cos\gamma\, \Sigma \frac{1+(-1)^{n+1}}{n} B_n \right]\Omega_y^2 + \left[\left(\frac{m_t}{3} + m_s\right)L^2 \sin\alpha\, \cos2\gamma\right.$$

$$+ \left(\frac{m_t}{2} + m_s\right)hL\sin\alpha\, \cos\alpha\, \cos\gamma + \frac{m_t h}{\pi} \cos2\gamma\, \Sigma \frac{1+(-1)^{n+1}}{n} C_n$$

$$\left. - \frac{m_t L}{\pi} \cos\alpha\, \cos\gamma\, \Sigma \frac{(-1)^n}{n} C_n \right] \Omega_y \Omega_z + \left[-2\left(\frac{m_t}{3} + m_s \right)L^2\cos\alpha\, \sin^2\gamma\right.$$

$$\left. + \frac{2m_t L}{\pi} \cos\alpha\, \sin 2\gamma\, \Sigma \frac{(-1)^n}{n} B_n \right]\Omega_x \dot{\alpha} + \left[\left(\frac{m_t}{3} + m_s\right)L^2 \sin 2\gamma\right.$$

$$+ 2\left(\frac{m_t}{2} + m_s\right)hL\, \cos\alpha\, \sin\gamma - \frac{2m_t h}{\pi}\cos\alpha\, \cos\gamma\, \Sigma \frac{1+(-1)^{n+1}}{n} B_n$$

$$\left. - \frac{2m_t L}{\pi} \cos2\gamma\, \Sigma \frac{(-1)^n}{n} B_n \right]\Omega_y\, \dot{\alpha} - \left(\frac{m_t}{3} + m_s\right)L^2 \sin\alpha\, \Omega_z\, \dot{\alpha}$$

$$+ \left(\frac{m_t}{2} + m_s\right)hL\sin\alpha\, \cos\gamma\, \Omega_y\, \dot{\gamma} - \left(\frac{m_t}{2} + m_s\right)hL\sin\gamma\, \Omega_z\, \dot{\gamma}$$

$$+ (\frac{m_t}{3} + m_s)L^2 \sin\gamma\cos\gamma\,\dot{\alpha}^2 + \frac{m_t L}{\pi}\cos\gamma\,\Sigma\frac{(-1)^n}{n}C_n\,\dot{\alpha}\,\dot{\gamma} - \frac{\rho}{\pi}\Sigma\frac{1+(-1)^n}{n}B_n\,\dot{L}^2$$

$$[-2\,(\frac{m_t}{2} + m_s)L\sin\alpha + \frac{m_t}{\pi}\cos\alpha\cos\gamma\,\Sigma\frac{1+3(-1)^n}{n}C_n\,]\Omega_x\dot{L}$$

$$+[\frac{m_t\sin\gamma}{\pi}\Sigma\frac{1+3(-1)^n}{n}C_n - \frac{\rho}{2}\Sigma B_nC_n\,]\Omega_y\dot{L} + \frac{\sin\gamma}{\pi}\Sigma\frac{1+(-1)^{n+1}}{n}C_n\,\dot{\alpha}\,\dot{L}$$

$$+ 2(\frac{m_t}{2} + m_s)L\,\dot{L}\,\dot{\gamma} - \frac{m_t}{\pi}\Sigma\frac{1+3(-1)^n}{n}\dot{B}_n\,\dot{L} + \frac{2\,m_t L}{\pi}\cos\alpha\cos\gamma\,\Sigma\frac{(-1)^n}{n}\dot{C}_n\,\Omega_x$$

$$+ \frac{2\,m_t L}{\pi}\sin\gamma\,\Sigma\frac{(-1)^n}{n}\dot{C}_n\,\Omega_y \quad = Q_\gamma \qquad (143)$$

Stretching equation

$$(m_t + m_s)\,\ddot{L} + \frac{m_t}{\pi}[\sin\alpha\,\Sigma\frac{1-3(-1)^n}{n}B_n - \cos\alpha\sin\gamma\,\Sigma\frac{1-3(-1)^n}{n}C_n\,]\,\dot{\Omega}_x$$

$$+[-(m_t + m_s)h\sin\alpha\cos\gamma + \frac{m_t}{\pi}\cos\gamma\,\Sigma\frac{1-3(-1)^n}{n}C_n - \frac{m_t}{\pi}(\frac{h}{L})\cos\alpha\,\Sigma\frac{1+(-1)^{n+1}}{n}C_n\,]\dot{\Omega}_y$$

$$+ [(m_t + m_s)h\sin\gamma - \frac{m_t}{\pi}\cos\alpha\,\Sigma\frac{1-3(-1)^n}{n}B_n + \frac{m_t}{\pi}(\frac{h}{L})\cos\gamma\,\Sigma\frac{1+(-1)^{n+1}}{n}B_n\,]\,\dot{\Omega}_z$$

$$+ \frac{m_t}{\pi}\cos\gamma\,\Sigma\frac{1-3(-1)^n}{n}C_n\,\ddot{\alpha} - \frac{m_t}{\pi}\Sigma\frac{1-3(-1)^n}{n}B_n\,\ddot{\gamma} + [(\frac{m_t}{2} + m_s)L\,(\cos^2\alpha\cos^2\gamma - 1)$$

$$+ \frac{2m_t}{\pi}\cos\alpha\sin\gamma\cos\gamma\,\Sigma\frac{(-1)^n}{n}B_n + \frac{2m_t}{\pi}\sin\alpha\cos\alpha\cos\gamma\,\Sigma\frac{(-1)^n}{n}C_n\,]\,\Omega_x^2 + [\,2(\frac{m_t}{2} + m_s)L$$

$$\cos\alpha\sin\gamma\cos\gamma + (\frac{m_t}{2} + m_s)h\sin\gamma - \frac{2m_t}{\pi}\cos\alpha\cos2\gamma\,\Sigma\frac{(-1)^n}{n}B_n + \frac{2m_t}{\pi}\sin\alpha\sin\gamma\,\Sigma\frac{(-1)^n}{n}C_n$$

$$+ \frac{\rho h}{\pi}\cos^2\alpha\cos\gamma\,\Sigma\frac{1+(-1)^{n+1}}{n}B_n\,]\Omega_x\Omega_y + [(\frac{m_t}{2} + m_s)L\sin2\alpha\cos^2\gamma$$

$$+(\frac{m_t}{2} + m_s)h\sin\alpha\cos\gamma + \frac{m_t}{\pi}\sin2\alpha\sin2\gamma\,\Sigma\frac{(-1)^n}{n}B_n - \frac{2m_t}{\pi}\cos2\alpha\cos\gamma\,\Sigma\frac{(-1)^n}{n}C_n$$

$$+ \frac{\rho h}{\pi}\cos\alpha\,\Sigma\frac{1+(-1)^{n+1}}{n}C_n\,]\Omega_x\Omega_z - [(\frac{m_t}{2} + m_s)L\cos^2\gamma + (\frac{m_t}{2} + m_s)h\cos\alpha\cos\gamma$$

$$+ \frac{\rho h^2}{2} + \frac{2m_t}{\pi}\sin\gamma\cos\gamma\,\Sigma\frac{(-1)^n}{n}B_n + \frac{\rho h}{\pi}\cos\alpha\sin\gamma\,\Sigma\frac{1+(-1)^{n+1}}{n}B_n\,]\Omega_y^2$$

$$+ [(\frac{m_t}{2} + m_s)L\sin\alpha\sin2\gamma - \frac{2m_t}{\pi}\sin\alpha\cos2\gamma\,\Sigma\frac{(-1)^n}{n}B_n$$

$$+ \frac{\rho h}{\pi}\sin\alpha\cos\alpha\cos\gamma\,\Sigma\frac{1+(-1)^{n+1}}{n}B_n - \frac{2m_t}{\pi}\cos\alpha\sin\gamma\,\Sigma\frac{(-1)^n}{n}C_n\,]\,\Omega_y\Omega_z$$

$$-[(\frac{m_t}{2}+m_s)L\cos\alpha^2+(\frac{m_t}{2}+m_s)h\cos\alpha\cos\gamma]\Omega_y^2-(\frac{m_t}{2}+m_s)L\cos\gamma^2\dot{\alpha}^2$$

$$-\frac{m_t}{\pi}\sin\gamma\sum\frac{1+(-1)^{n+1}}{n}B_n\dot{\alpha}\dot{\gamma}-(\frac{m_t}{2}+m_s)L\dot{\gamma}^2+2(\frac{m_t}{2}+m_s)L\cos\alpha\sin\gamma\cos\gamma\,\Omega_x\dot{\alpha}$$

$$+[2(\frac{m_t}{2}+m_s)L\sin\alpha-\frac{m_t}{\pi}\cos\alpha\cos\gamma\sum\frac{1+(-1)^{n+1}}{n}C_n]\Omega_x\dot{\gamma}$$

$$-[2(\frac{m_t}{2}+m_s)L\cos\gamma^2+(\frac{3m_t}{2}+2m_s)h\cos\alpha\cos\gamma]\Omega_y\dot{\alpha}+[\frac{m_th}{2}\sin\alpha\sin\gamma$$

$$+\frac{m_th}{\pi L}\sin\alpha\cos\gamma\sum\frac{1+(-1)^{n+1}}{n}B_n-\frac{m_t}{\pi}\sin\gamma\sum\frac{1+(-1)^{n+1}}{n}C_n]\Omega_y\dot{\gamma}$$

$$+[(\frac{m_t}{2}+m_s)L\sin\alpha\sin2\gamma-\frac{2m_t}{\pi}\sin\alpha\cos2\gamma\sum\frac{(-1)^n}{n}B_n$$

$$-\frac{m_t}{\pi}\cos\alpha\sin\gamma\sum\frac{1+(-1)^{n+1}}{n}C_n]\Omega_z\dot{\alpha}+[\frac{m_th}{2}\cos\gamma-2(\frac{m_t}{2}+m_s)L\cos\alpha$$

$$-\frac{m_th}{\pi L}\sin\gamma\sum\frac{1+(-1)^{n+1}}{n}B_n-\frac{m_t}{\pi}\sin\alpha\cos\gamma\sum\frac{1+(-1)^{n+1}}{n}C_n]\Omega_z\dot{\gamma}+\rho\dot{L}^2$$

$$-\frac{m_t}{\pi}\cos\alpha\sum\frac{1+(-1)^{n+1}}{n}\dot{B}_n\Omega_z-\frac{m_t}{\pi}\sum\frac{1+(-1)^{n+1}}{n}\dot{B}_n\dot{\gamma}+\frac{m_t}{\pi}\cos\gamma\sum\frac{1+(-1)^{n+1}}{n}\dot{C}_n\Omega_y$$

$$+\frac{m_t}{\pi}\cos\gamma\sum\frac{1+(-1)^{n+1}}{n}\dot{C}_n\dot{\alpha}=T+Q_L \qquad (144)$$

4.4.3 Mode Equations of the Tether

The vibrating modes are coupled with the other degrees of freedom. They satisfy the following equations:

Out-of-plane mode

$$\frac{m_t}{2}\ddot{B}_n+\frac{\rho B_n}{4}\ddot{L}+\frac{m_th}{\pi}\frac{[1-(-1)^n]}{n}[\sin\alpha\sin\gamma\dot{\Omega}_x+\cos\gamma\dot{\Omega}_z]-\frac{m_t}{2}C_n\dot{\omega}_x$$

$$-\frac{m_tL}{\pi}\frac{(-1)^n}{n}\dot{\omega}_z-\frac{m_t}{2}(B_n\omega_x^2+\frac{2L}{\pi}\frac{(-1)^n}{n}\omega_x\omega_z+B_n\omega_z^2-C_n\omega_y\omega_z+2C_n\omega_x)$$

$$+\frac{m_th}{\pi}\frac{[1-(-1)^n]}{n}(\cos\alpha\,\omega_x\Omega_y-\sin\alpha\cos\gamma\,\omega_z\Omega_y+\sin\gamma\,\omega_z\Omega_z+\cos\alpha\sin\gamma\,\dot{\alpha}\Omega$$

$$+\sin\alpha\cos\gamma\,\dot{\gamma}\,\Omega_y-\sin\gamma\,\dot{\gamma}\,\Omega_z)+\frac{\rho\dot{L}}{2}(\dot{B}_n-C_n\omega_x)+\frac{m_th}{\pi}\frac{[1-(-1)^n]}{n}(\dot{\gamma}+$$

$$\cos\alpha\,\Omega_z-\sin\alpha\,\Omega_x)\dot{L}+\frac{\pi^4EI}{2L^3}B_n=0 \qquad (145)$$

In–plane mode

$$\frac{m_t}{2}\ddot{C}_n + \frac{\rho C_n}{4}\ddot{L} - \frac{m_t h}{\pi}\frac{[1-(-1)^n]}{n}\cos\alpha\,\dot{\Omega}_y + \frac{m_t}{2}B_n\dot{\omega}_z + \frac{m_t L}{\pi}\frac{(-1)^n}{n}\dot{\omega}_y$$

$$- \frac{m_t}{2}\left(C_n \omega_x^2 + C_n \omega_y^2 + \frac{2L}{\pi}\frac{(-1)^n}{n}\omega_x\omega_z - B_n\omega_y\omega_z - \dot{B}_n\omega_x \right) + \frac{m_t h}{\pi}\frac{[1-(-1)^n]}{n}$$

$$\left(\sin\alpha\,\dot{\alpha}\,\Omega_y + \sin\alpha\cos\gamma\,\omega_y\Omega_y - \sin\gamma\,\omega_y\Omega_z + \sin\alpha\sin\gamma\,\omega_x\Omega_y + \cos\gamma\,\omega_x\Omega_z \right)$$

$$- \frac{m_t h}{\pi}\frac{[1-(-1)^n]}{n}\omega_y\dot{L} + \frac{\rho B_n}{2}\omega_x\dot{L} + \frac{\rho\dot{C}_n\dot{L}}{2} + \frac{\pi^4 EI}{2L^3}C_n = 0 \qquad (146)$$

4.5 Conclusion

The nonlinear dynamical equations of the tethered antenna system have been obtained. We can see that all the degrees of freedom are coupled in the equations. From Eq. (142) and Eq. (143), it is seen that the in–plane and out–of–plane motions are coupled through second–order, and also coupled with the flexibility of the tether. The dynamical behavior of such a complex system (including altitude motions of the satellite and flexibility of the tether) has never been studied before. Next step of our research will concentrate on the simulation of the dynamical behavior of the system during the deployment and retrieval.

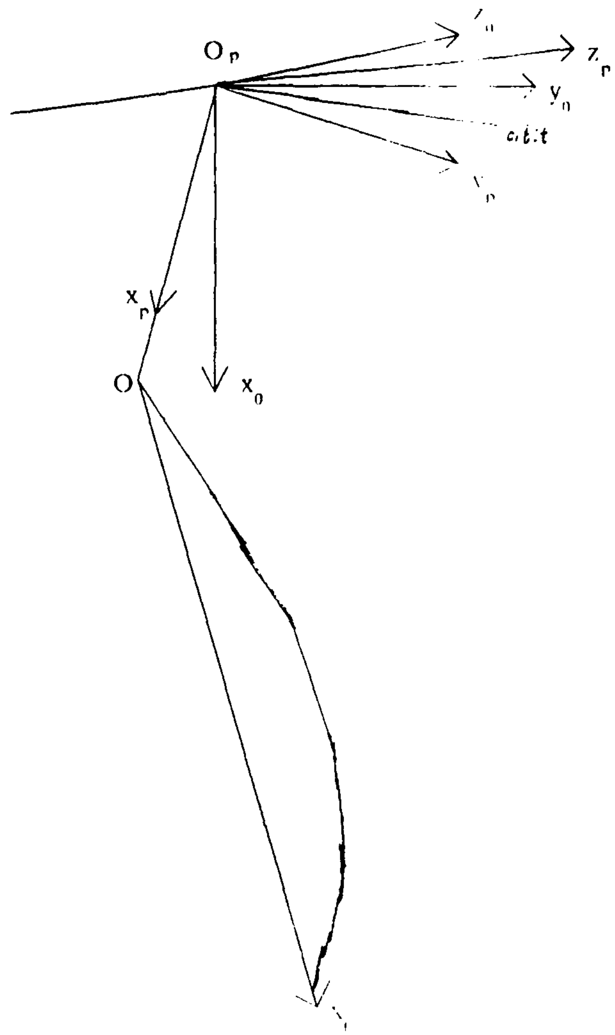


Fig. 7 Coordinate systems used in the development of nonlinear dynamic equations

$O_p X_0 Y_0 Z_0$ orbital-fixed reference frame

$O_p X_p Y_p Z_p$ undeformed shell reference frame, R_p, with unit vectors $\hat{i}_p$, $\hat{j}_p$, $\hat{k}_p$

$O\ X_t\ Y_t\ Z_t$ subsatellite-undeformed tether frame, R_t, with unit vectors $\hat{i}_t$, $\hat{j}_t$, $\hat{k}_t$

Some integrals used in this development

$$(1)\quad \int_0^L v\,dx = \int_0^L \Sigma \sin\frac{n\pi x}{L} B_n\,dx = \frac{L}{\pi}\Sigma \frac{1+(-1)^{n+1}}{n} B_n$$

$$(2)\quad \int_0^L w\,dx = \int_0^L \Sigma \sin\frac{n\pi x}{L} C_n\,dx = \frac{L}{\pi}\Sigma \frac{1+(-1)^{n+1}}{n} C_n$$

$$(3)\quad \int_0^L xv\,dx = \int_0^L x\,\Sigma \sin\frac{n\pi x}{L} B_n dx = -\frac{L^2}{\pi}\Sigma \frac{(-1)^n}{n} B_n$$

$$(4)\quad \int_0^L xw\,dx = \int_0^L x\,\Sigma \sin\frac{n\pi x}{L} C_n\,dx = -\frac{L^2}{\pi}\Sigma \frac{(-1)^n}{n} C_n$$

$$(5)\quad \int_0^L vw\,dx = \frac{L}{2}\Sigma B_n C_n$$

$$(6)\quad \int_0^L v^2 dx = \frac{L}{2}\Sigma B_n^2$$

$$(7)\quad \int_0^L w^2 dx = \frac{L}{2}\Sigma C_n^2$$

$$(8)\quad \int_0^L x \cos\frac{n\pi x}{L}\,dx = \frac{L^2}{n^2\pi^2}\left[(-1)^n-1\right]$$

5. CONCLUSIONS AND RECOMMENDATIONS

The system linear equations for the motion of a tethered shallow spherical shell in orbit with its symmetry axis nominally following the local vertical are developed. The shell roll, yaw tether out-of-plane swing motion and elastic vibrations are decoupled from the shell and tether in-plane pitch motions and elastic vibrations. The neutral gravity stability conditions for the special case of a constant length rigid tether are given for in-plane motion and out-of-plane motion. It is proved that the in-plane motion of the system could be asymptotically stable based on Rupp's tension control law, for a variable length tether. However, the sytem simulation results indicate that the transient responses can be improved significantly, especially for the damping of the tether and shell pitch motion, by an optimal feedback control law for the rigid variable length tether model. It is also seen that the system could be unstable when the effect of tether flexibility is included if the control gains are not chosen carefully. The transient responses for three different tension control laws are compared during typical station keeping operations. The transient responses can be further improved by including the state feedback of the tether vibrational modes into the optimal control law, especially for the damping of the tether vibrations.

Extensions to the present study could consider the effect of the shell flexibility on the system stability and control and some kind of active control could be introduced (in addition to tether tension control) to improve system performance. Additional control will be required to provide for out-of-plane damping of rigid motions and vibration suppression.

Because most of the useful missions are carried out during the station keeping phase for Shuttle/Platform Tethered Subsatellite systems, a review of the various tether system control laws has focused mainly on the deployment and station keeping stages. Retrieval is less important than the first two for tether reflector applications where it may not be required to retrieve the tether (except possibly before rapid maneuvering). Although some nonlinear control laws were proposed (especially those which include thruster augmentation of the tether tension control), it may still be difficult to implement an efficient and successful retrieval for the tethered reflector system and further study is suggested.

For deployment, the active tension modulation schemes proposed by Rupp and improved by subsequent investigators are more efficient than the purely passive scheme advocated by Kane and Levinson. A tension control law, based on a combination of exponential and uniform rate of tether length, as a commanded length rate can be used both for in-plane and out-of-plane motion control during deployment.

Out of the various active control laws for proposed station keeping of orbiting tethered reflector systems, Rupp's control law is the most basic and is effective in controlling in-plane motion, but not adequate for out-of-plane motion control. Control laws based on optimal control theory offer the greatest potential for applications involving proposed orbiting tethered reflector systems. Alternate tension modulation optimal control laws based on both in-plane tether swing angle and vibrational state information can result in further improvements as compared with Rupp's control law.

For out-of-plane motion control during station keeping a combination of tension modulation in the tether plus other forms of control (such as the use of thrusters) will be required.

Finally , a preliminary model of the nonlinear dynamics of the orbiting tethered antenna/reflector system has been developed based on Lagrangian formulation. The resulting equations are highly coupled and for deployment represent a set of non-autonomous differential equations. For this model the shell was considered to be rigid, but the mass and flexibility of the tether has been taken into account. These equations will be used in the next phase of this effort to simulate deployment dynamics and compare the performance using different control strategies.

References

1. Yasaka, Tetsuo, "The Structural Feasibility of a Gravity Stabilized Antenna," Acta Astronautica, Vol. 8, No. 5-6, 1981, pp. 689-701.

2. Kumar, V.K. and Bainum, P.M, "Motion of a Flexible Shallow Spherical Shell in Orbit," AIAA Journal, Vol. 20, No. 8, August, 1982, pp. 1113-1119.

3. Bainum, P.M. and Reddy, A.S.S.R., "On the Shape and Orientation Control of an Orbiting Shallow Spherical Shell Structure," IFAC/ESA Symposium on Automatic Control in Space, Noordwijkerhout, The Netherlands, July 5-9, 1982.

4. Bainum, P.M., Woodard, S.E., and Juang, J.-N., "The Development of Optimal Control Laws for Orbiting Tethered Platform Systems," AAS/AIAA Astrodynamics Specialist Conference, Vail, Colorado, August 12-15, 1985, Paper No. AAS 85-360; also the Journal of the Astronautical Sciences, Vol. 35, No. 2, April-June 1987, pp. 135-153.

5. Fan, Ruying and Bainum, P.M., "The Dynamics and Control of a Space Platform Connected to a Tethered Subsatellite," AIAA/NASA/PSN International Conference on Tethers in Space, Arlington, Va., September 17-19, 1986; also the Journal of Guidance, Control and Dynamics, Vol. 11, No. 4, July-August, 1988, pp. 377-381.

6. Santini, P., "Stability of Flexible Spacecrafts," Acta Astronautica, Vol. 3, 1976, pp. 685-713.

7. Bainum, P.M., Kumar, V.K., and James, P.K., "The Dynamics and Control of Large Flexible Space Structures, Part B: Development of Continuum Model and Computer Simulation," NASA-CR-156976, May 1978.

8. Liu, Liangdong and Bainum, P.M., "Effect of Tether Flexibility on the Tethered Shuttle Subsatellite Stability and Control," PSN/NASA/AIDAA Second International Conference on Tethers in Space, Venice, Italy, October 4-8, 1987; also the Journal of Guidance, Control and Dynamics, Vol. 12, No. 6, Nov.-Dec., 1989, pp. 866-873.

9. Johnson, M.W. and Reissner, E., "On Transverse Vibrations of Shallow Spherical Shells," Quarterly Journal of Applied Mathematics, Vol. 15, No. 4, Jan. 1958, pp. 367-380.

10. Rupp, C.C., "A Tether Tension Control Law for Tethered

Subsatellites Deployed along the Local Vertical," NASA-TMX-64963, 1985.

11. Misra, A.K., Modi, V.J., "Dynamics and Control of Tethered Connected Two-Body Systems - A Brief Review," 33rd International Astronautical Congress, Paris, Sept. 27 - Oct. 2, 1982, Paper No. IAF82-315.

12. Baker, W.P., Dunken, J.A., Galaboff, Z.J., Johnston, K.D., Kissel, R.R., Rheinfurth, M.H. and Siebel, M.P.L. "Tethered Subsatellite Study," NASA TMX-73314, Marshall Space Flight Center, Alabama, 1976.

13. Kalaghan, P.N., Arnold, D.A., Colombo, G., Grossi, M.D. Korschner, L.R. and Orringer, O., "Study of the Dynamics of a Tethered Satellite System (Skyhook)," Final Report, Contract NASA 832199, Smithsonian Institution, Astrophysical Observatory, Cambridge, Massachusetts, 1978.

14. Misra, A.K. and Modi, V.J., "Deployment and Retrieval of a Subsatellite connected by a Tether to the Space Shuttle," AIAA/AAS Astrodynamics Conference, Danvers, Massachusetts, August 1980, Paper No. AIAA-80-1693.

15. Bainum, P.M., and Kumar, V.K., "Optimal Control of the Shuttle-Tethered-System," Acta Astronautica, Vol. 7, 1980, pp. 1333-1348.

16. Xu, D.M., Misra, A.K. and Modi, V.J., "Three Dimensional Control of the Shuttle Supported Tethered Satellite Systems during Deployment and Retrieval," Proceedings of the 3rd VPISU/AIAA Symposium on Dynamics and Control of Large Flexible Spacecraft, Blacksburg, Virginia, 1981.

17. Modi, V.J., Chang-fu, G., Misra, A.K. and Xu, D.M., "On the Control of the Space Shuttle Based Tethered Systems," Acta Astronautica, Vol. 9, No. 6-7, 1982, pp. 437-443.

18. Boschitsch, Alexander and Bendiksen, Oddvar O., "Nonlinear Control Laws for Tethered Satellites," Advances in the Astronautical Sciences, Vol. 62, Tethers in Space, 1986, pp. 257-276.

19. Banerjee, A.K. and Kane, T.R., "Tethered Satellite Retrieval with Thruster Augmented Control," AIAA/AAS Astrodynamics Conference, San Diego, California, August 1982, Paper No. AIAA-82-1421.

20. Misra, A.K. and Modi, V.J., "A General Dynamical Model for the Space Shuttle Based Tethered Subsatellite System," Advances in the Astronautical Sciences, Vol. 40, Part II, 1979, pp.537-557.